Department of Mathematics
Year 13 Scheme of Work – Pure

$3 - 2 = 1$ → $\sin^2\theta + \cos^2\theta = 1$ → $\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{-\frac{1}{2}t^2}dt = 1$

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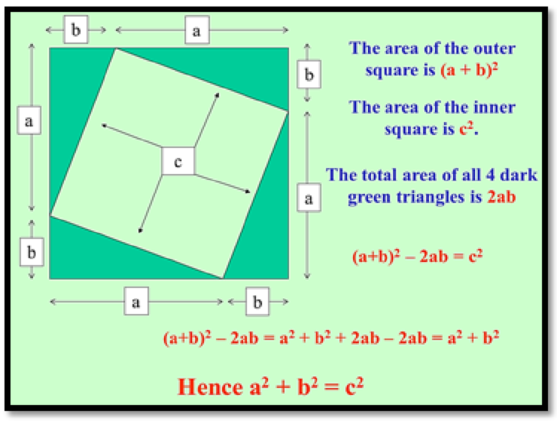
CHAPTER 1 – Algebraic Methods

Chapter 1 – Algebraic Methods (Part A – Proof)

By the end of the sub-unit, students should:

- understand that various types of proof can be used to give confirmation that previously learnt formulae are true, and have a sound mathematical basis;
- understand that there are different types of proof and disproof (e.g. deduction and contradiction), and know when it is appropriate to use which particular method;
- be able to use an appropriate proof within other areas of the specification later in the course.

Introduce using areas and the expansion of $(a+b)^2$ to prove Pythagoras' theorem as an example of using a logical sequence of steps in order to deduce a familiar result.



Explain how verification for a set number of values in *not* a proof of a general result (for all values of n).

Show how different methods can be used to prove a statement, including:

- Manipulating the LHS of a result and using logical steps (normally algebraic) to make it match the RHS or vice versa (or, sometimes, manipulating both sides to reach the same expression).
- Manipulating an expression to show it holds true for all values. For example, an inequality can always be ≥ 0 if we manipulate the LHS to be in the form of $[\text{something}]^2$ since anything squared will always be bigger or equal to zero. This argument can be used on a gradient function to prove a function is increasing.

Provide standard examples of proof by contradiction, e.g., $\sqrt{2}$ is irrational:

- Assuming it can be written as a rational number $\frac{a}{b}$ which has been written in its lowest terms.
- It follows that $\frac{a^2}{b^2} = 2$ and $a^2 = 2b^2$. Therefore, a^2 is even because it is equal to $2b^2$.
- It follows that a must be even (as squares of odd integers are never even).
- Because a is even, there exists an integer k that fulfills: $a = 2k$.
- Substituting $2k$ for a above gives $2b^2 = (2k)^2 = 4k^2$, so $b^2 = 2k^2$.
- Because $2k^2 = b^2$, it follows that b^2 is even and b is also even.
- Hence a and b are both even, which contradicts that $\frac{a}{b}$ is in its simplest form

Another example of proof by contradiction is the proof that there is an infinite number of primes:

- Assume there is an integer p, such that p is the largest prime number.
- Now $p! + 1 > p$ and is not divisible by p or any other number less than p *
  *If 2 is a factor of n, then 2 is not a factor of $n + 1$. Similarly if 3 is a factor of n, 3 is not a factor of $n + 1$.
  Now 2, 3, … p are all factors of $p!$, so none are factors of $p! + 1$.
- So, either $p! + 1$ is not divisible by an integer other than 1 or $p! + 1$ which means $p! + 1$ is prime, or $p! + 1$ is divisible by some number between p and $p! + 1$ which implies there is a prime number larger than p.
- These both contradict our initial assumptipn, which proves there are an infitie number of primes.

Illustrate proof by exhaustion e.g. Prove that $1^3 + 2^3 + 3^3 + \ldots + n^3 = (1 + 2 + 3 + \ldots + n)^2$ for the positive integers from 1 to 5 inclusive.

- This can be proved if you substitute (exhaust) all the possible values of n from 1 to 5. Note that this type of proof can only be used for proving something for a set of given values.

You should also talk about disproof by counter-example.

Explain that all we have to do is find *one* example where the statement does not hold and this is enough to show that it is not always true. This method can be used to disprove trigonometric identities as well as statements such as $a > b \Rightarrow a^2 > b^2$:

- Choose any pair of negative numbers with $a > b$ e.g. $a = -2$ and $b = -3$.
- Hence $a > b$, but if we square the numbers $a^2 < b^2$ (as $4 < 9$) and so this disproves the statement.

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Chapter 1 – Algebraic Methods (Part B – Algebraic Fractions) By the end of the sub-unit, students should: • be able to add, subtract, multiply and divide algebraic fractions; • know how to use the factor theorem to shown a linear expression of the form $(a + bx)$ is a factor of a polynomial; • know how to use the factor theorem for divisors of the form $(a + bx)$; • be able to simplify algebraic fractions by fully factorising polynomials up to cubic.	Revise the basic rules of numerical fractions and start with simplifying some GCSE (9-1) Mathematics algebraic fractions. Exam questions tend to focus on factorising polynomials and then cancelling common factors to simplify algebraic fractions. For example: Simplify $\frac{x^2-5x-6}{x^2-10x+24} \div \frac{x^2-x-2}{x^2-4x}$ You can use function notation when referring to fractions. (This has been covered in GCSE (9-1) Mathematics and also links with Unit 3.) For example: The function f is defined by $\mathrm{f}: x \to \frac{3(x+1)}{2x^2+7x-4} - \frac{1}{x+4}, x \in \mathbb{R}, x > \frac{1}{2}$ Show that $\mathrm{f}(x) = \frac{1}{2x-1}$
Chapter 1 – Algebraic Methods (Part C – Partial Fractions) By the end of the sub-unit, students should: • be able to split a proper fraction into partial fractions; • be able to split an improper fraction into partial fractions, dividing the numerator by the denominator (by polynomial long division or by inspection).	Stress the fact that when we break-up a fraction into two or more partial fractions, we use an identity (≡) sign, and not an equal sign, as the expressions are equivalent for all values of x. Start with a pair of algebraic fractions that need to be added together. Stress that the single fraction answer may be simplified, but that it can often be difficult to work with. For example in order to integrate the fraction it may be necessary to split it back up into two (or more) partial fractions. In other words, the reverse process from the previous section above needs to be carried out. The number of partial fractions and the format of the individual terms, is dependent on two factors. 1. The maximum power (or degree) of the polynomials of the numerator and denominator. The degree of the denominator must be *greater* than that of the numerator. If the degree is equal or the degree of the numerator is greater (i.e. the fraction is improper), then algebraic division must be carried out first, and then the partial fractions formed. 2. The type and power of denominator. If the denominator is, e.g. $(x+2)^2$, then we call this a *repeated* factor. In order to cover all possibilities of factors this has to be set up as two partial fractions with denominators $(x+2)$ and $(x+2)^2$. Show a numerical example with a denominator of 25, and hence the denominators of the partial fractions are 5 and 25.) Examples of each of the following types need to be covered. Linear: $\frac{5x-5}{(x+3)(x-2)}$ $\frac{2}{x^2-1}$ $\frac{7x+3}{x(x+1)}$ Repeated: $\frac{4x^2-3x+5}{(x-1)^2(x+2)} \equiv \frac{A}{(x-1)^2} + \frac{B}{(x-1)} + \frac{C}{(x+2)}$ Improper: $\frac{2x^2+5x-6}{(2x-1)(1+x)} \equiv A + \frac{B}{2x-1} + \frac{C}{1+x}$ As students work through examples, encourage them to experiment with the choice of values they substitute. If necessary remind them that $x = 0$ is an option. Also show that equating coefficients can sometimes be a more efficient alternative, sometimes avoiding the necessity for simultaneous equations.

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CHAPTER 2 – Functions and Modelling

Chapter 2 – Functions and Modelling (Part A – Modulus Functions) By the end of the sub-unit, students should: • understand what is meant by a modulus of a linear function; • be able to sketch graphs of functions involving modulus functions; • be able to solve equations and inequalities involving modulus functions.	Define the modulus of a set of numbers as being the positive values only. e.g. $\|-2\| = 2$ and $\|5\| = 5$. Begin by using an ICT graph-drawing package (either using the whiteboard or students' individual devices) to sketch some linear graphs using both $y =$ and $\mathrm{f}(x) =$ notation, e.g. $y = 2x - 1$ or $\mathrm{f}(x) = 2x - 1$. Display the graph of $y = \|2x - 1\|$ and discuss this with students, drawing comparisons with the 'non-modulus' graph and making sure everyone recognises that $y = \|2x - 1\|$ does not have any negative values of y (the graph 'bounces up' with the x-axis acting like a mirror). Define the term modulus function and use the general notation $y = \|\mathrm{f}x\|$. Ask students to predict what the graph of $y = \|2x\| - 1$ will look like and then plot it. This time the values of x that are substituted into the function cannot be negative. In other words the graph on the left of the y-axis is a reflection of the graph on the right (where the x-values are positive) with the y-axis being the line of symmetry. The general notation for this type of function is $y = \|\mathrm{f}x\|$. Students should be able to sketch the graphs of $y = \|ax + b\|$ and use their graphs to solve modulus equations and inequalities. Use the graph-drawing package to sketch the graph of $y = \|2x - 1\|$ and $y = x$ and use these to solve $\|2x - 1\| = x$ by considering the points of intersection. Ask students to think about how they might solve this equation algebraically without using a graph. Solving $\|2x - 1\| = x$ gives one solution, but how would the 'modulus' part be represented algebraically? What is the equation of the straight-line graph that represents the 'bounced' part which is now above the y-axis? Extend this idea to looking at inequalities, for example how to solve $\|2x - 1\| > x$.
Chapter 2 – Functions and Modelling (Part B – Composite and Inverse Functions) By the end of the sub-unit, students should: • be able to work out the domain and range of functions; • know the definition of a one-one and a many-one mappings; • be able to work out the composition of two functions; • be able to work out the inverse of a function and sketch its graph; • understand the condition for an inverse function to exist.	The notation f: $x \mapsto$ ... and $\mathrm{f}(x)$ will be used as in GCSE (9-1) Mathematics. Students will need to understand exactly what functions are and the notation associated with them. Domain and range from $\mathbb{R}$ (or a subset of $\mathbb{R}$) to $\mathbb{R}$ are important terms for students to understand and should be used regularly. Link this to function machines and graphs (where the domain is the set of x-values and the range is the set of corresponding y-values). Students should be aware of one-one and many-one mappings and know that a function cannot be one-many. Definitions and examples of odd and even functions will need to be given. Students need to know how to find the inverse of a function and it is worth stressing the notation here as lots of students still differentiate when they see this in an exam. Students should know that if f^{-1} exists, then $\mathrm{ff}^{-1} = \mathrm{f}^{-1}\mathrm{f}(x) = x$. It follows from this that the inverse of a many-one function can only exist if its domain is restricted to make it a one-one function. Composite functions are also introduced here and it is worth spending some time going over why the order is very important. Students must know that fg means 'do g first and then f''. It may be helpful to use an additional set of brackets in the notation for composite functions, e.g. $\mathrm{f}[\mathrm{g}(x)]$. Draw lots of examples of the above using graphing packages and relate the mappings to the graphs. Give an example of a quadratic in which the range is determined by the minimum or maximum point. Students must also know that the graph of $\mathrm{f}^{-1}(x)$ is the image of the graph of $y = \mathrm{f}(x)$ after reflection in the line $y = x$. You could relate this to the reverse function machine and the algebraic approach for finding an inverse function (when you change the subject of the formula and rewrite it in terms of x as the final step). Ask questions such as: When does the function machine fail to find an inverse? Do any functions have a self-inverse? Is an inverse function always possible?

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Chapter 2 – Functions and Modelling (Part C – Transformations) By the end of the sub-unit, students should: • understand the effect of simple transformations on the graph of $y = \mathrm{f}(x)$ including sketching associated graphs and *combinations* of the transformations: $y = a\mathrm{f}(x)$, $y = \mathrm{f}(x) + a$, $y = \mathrm{f}(x + a)$, $y = \mathrm{f}(ax)$ • be able to transform graphs to produce other graphs; • understand the effect of composite transformations on equations of curves and be able to describe them geometrically.	Students should have some understanding of graph transformations from GCSE (9-1) Mathematics and AS Mathematics - Pure Mathematics, but this will not necessarily include combinations of transformations. Students need to be able to sketch the transformations $y = a\mathrm{f}(x) + b$, $a\mathrm{f}(x + b)$ and $\mathrm{f}(ax) + b$, but will not be required to sketch $\mathrm{f}(ax + b)$ Use graph drawing packages to investigate the properties of familiar functions (such as trigonometric and exponential functions) when you apply the above transformations. Relate the geometry of the transformation to the algebra. For example, $\mathrm{f}(x) + a$ adds a to all the y-coordinates, hence the graph moves 'up' by a units (translation vector). Pose the question, "Does the order in which transformations are applied matter?" Ask students to explore this and present their findings to the class.
Chapter 2 – Functions and Modelling (Part D – Modelling with Functions) By the end of the sub-unit, students should: • use functions in modelling, including consideration of limitations and refinements of the models.	The specification gives some possible contexts in which functions can be used to model real-life situations. These are: • Use of trigonometric functions for modelling tides, hours of sunlight, etc. • (See the example in Unit 6g) • Use of exponential functions for growth and decay (See AS Mathematics content - Pure Mathematics, Section 6.7). • Use of reciprocal function for inverse proportion (e.g. Pressure and volume)

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Chapter 3 – Sequences and Series

Chapter 3 – Sequences and Series (Part A – Arithmetic and Geometric Progression) By the end of the sub-unit, students should: • know what a sequence of numbers is and the meaning of finite and infinite sequences; • know what a series is; • know the difference between convergent and divergent sequences; • know what is meant by arithmetic series and sequences; • be able to use the standard formulae associated with arithmetic series and sequences; • know what is meant by geometric series and sequences; • be able to use the standard formulae associated with geometric series and sequences; • know the condition for a geometric series to be convergent and be able to find its sum to infinity; • be able to solve problems involving arithmetic and geometric series and sequences; • know the proofs and derivations of the sum formulae (for both AP and GP).	Start by recapping the work students did on sequences at GCSE (9-1) Mathematics before moving on to the new A level content, paving the way for the sigma notation in the following section. Use practical situations, for example involving money, to illustrate APs and GPs and contrast the different ways they grow. Find the *n*th term of a given arithmetic sequences and also use the rule to find the next two terms. The Gauss problem (1 + 2 + … + 1000) is a good numerical way to lead into the full proof of the sum of an AP. Students will need to know the proof and derivation of the formula for the sum of an arithmetic sequence. Illustrate how arithmetic sequences are different to geometric sequences, and explain that the common difference (a) becomes the common ratio (r). Students need to be aware that not all geometric sequences converge. Cover problems where the n in the *n*th term formula ar^{n-1} is to be found using logarithms. (Show that it works if we use either base 10 or e.) Illustrate when to use $\frac{a(1-r^n)}{(1-r)}$ and when to use $\frac{a(r^n-1)}{(r-1)}$ (depending on the value of r). Show that $\frac{a}{(1-r)}$ can be derived if we illustrate on a calculator that r^n tends to zero when $-1 < r < 1$. A way of illustrating the sum to infinity is to imagine hammering in a nail into a piece of wood, where each strike makes the nail sink in exactly half its remaining distance. There will be a limit to how many times it will need to be hit, as it surely will end up being 'flush' to the surface of the wood and have a distance of zero above the wood. (You can link this to Zeno's paradox.)
Chapter 3 – Sequences and Series (Part B –Sigma Notation) By the end of the sub-unit, students should: • be familiar with $\sum$ notation and how it can be used to generate a sequence and series; • know how this notation will lead to an AP or GP and its sum; • know that $$\sum_{1}^{n} 1 = n$$	The key to understanding the concept of $\sum$ is to look at the limit values and substitute them into the nth term formula to generate the terms of the sequence. Emphasise to students that they must take care when finding the starting point and never assume it starts with $n = 1$. Students may initially find the $\sum$ notation tricky, particularly if they are not asked

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Chapter 3 – Sequences and Series (Part C – Recurrence & Iterations) By the end of the sub-unit, students should: • know that a sequence can be generated using a formula for the *n*th term or a recurrence relation of the form $x_{n+1} = \mathrm{f}(x_n)$; • know the difference between increasing, decreasing and periodic sequences; • understand how a recurrence relation of the form $U_n = \mathrm{f}(U_{n-1})$ can generate a sequence; • be able to describe increasing, decreasing and periodic sequences.	Work with sequences including those given by a formula for the *n*th term and those generated by a simple relation of the form $x_{n+1} = \mathrm{f}(x_n)$ and link this with the work done on iterations in GCSE (9-1) Mathematics. Explore $x_{n+1} = \mathrm{f}(x_n)$ type series using graphics calculators or spreadsheets. (You can draw links between this work and Unit 9 - Numerical methods.) Move on to general recurrence relations of the form $U_n = \mathrm{f}(U_{n-1})$ and investigate which sequences converge, diverge and oscillate. Spend some time looking at the different forms of notation for recurrence relations, making sure you cover examples of increasing, decreasing and periodic sequences. For example, $u_n = \frac{1}{3n+1}$ describes a decreasing sequence as $u_{n+1} < u_n$ for all integers n $u_n = 2^n$ is an increasing sequence as $u_{n+1} > u_n$ for all integers n $u_{n+1} = \frac{1}{u_n}$ for $n > 1$ and $u_1 = 3$ describes a periodic sequence of order 2.

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Chapter 4 – Binomial Expansion

Chapter 4 – Binomial Expansion (Part A – Expansion of Functions) By the end of the sub-unit, students should: • be able to find the binomial expansion of $(1-x)^{-1}$ for rational values of n and $\|x\| < 1$; • be able to find the binomial expansion of $(1+x)^n$ for rational values of n and $\|x\| < 1$; • be able to find the binomial expansion of $(1+bx)^n$ for rational values of n and $\|x\| < \frac{1}{\|b\|}$; • be able to find the binomial expansion of $(a+x)^n$ for rational values of n and $\|x\| < a$; • be able to find the binomial expansion of $(a+bx)^n$ for rational values of n and $\|\frac{bx}{a}\| < 1$; • know how to use the binomial theorem to find approximations (including roots).	Begin by reviewing the expansion of $(a+b)^n$ when n is a positive integer. Ask students to expand $(1+x)^4$ and then try $(1+x)^{-2}$. Why does it fail to work? Which coefficient calculation breaks down? Explain how the binomial theorem allows us to expand *any* power. (Explain the reasoning behind the factorial notation using the explanation in the *Reasoning and problem solving* section below.) Consider why the expansions are infinite when the power is *not* a positive integer. How far do we need to expand and to which term? (For example, up to and including coefficients of x^3.) Take care to show the precision needed when dealing with negative calculations by demonstrating examples such as $(1-2x)^{-\frac{1}{2}}$. If we expanded $(1+x)^{\frac{1}{2}}$ then substituted $x = -0.1$, we would be effectively finding the square root of 0.9. Ask students to use a calculator to find an accurate value for $\sqrt{9}$. How many terms of the expansion would we need to substitute into in order to get a 4 decimal place version of the accurate value? What happens when we substitute $x = 3$? Does this find the square root of 4? Explain that if we raise a number > 1 to a positive power, it 'grows' and diverges out of control. This means that the value of x must be such that $-1 < x < 1$ or $\|x\| < 1$ in order to use the expansion of $(1+x)^n$. The validity of the expansion is dependent upon the value of x we substitute into the terms. Cover examples that build-up the expansions listed in the objectives above, ending with $(a+bx)^n$ for rational values of n and valid for $\frac{\|bx\|}{a} < 1$. Introduce the concept of expansions of expressions which start with a rather than 1. Begin by showing that if we have $(2+x)$ and if we want to make this start with a 1 in the bracket, we must take out the factor of 2, giving $2(1+\frac{x}{2})$. Now show for example, that $2(1+4)$ gives the same result as $(2+8)$ if we multiplied this out, but that if the bracket were squared the result would not be the same i.e. $2(1+4)^2 \neq (4+8)^2$.. However, $2^2(1+4)^2 = (4+8)^2$, so we need to raise the factor to the same power of the bracket and $(a+bx)^n = a^n(1+\frac{bx}{a})^n$.
Chapter 4 – Binomial Expansion (Part B – Using Partial Fractions) By the end of the sub-unit, students should: • be able to use partial fractions to write a rational function as a series expansion.	This sub-unit links with sub-unit 2b above (Partial Fractions) and gives the students a purpose for learning how to break-up a rational function into two or more partial fractions. If we consider the 'complicated' fraction below, it needs to be simplifies into two simpler fractions each of which only involve a *single* algebraic bracket. $\frac{2x^2+5x-10}{(x-1)(x+2)} \equiv A + \frac{B}{x-1} + \frac{C}{x+2}$ We can now rewrite each term as a binomial series. (It is important to demonstrate that the $\frac{B}{x-1}$ term will become $B(x-1)^{-1}$.) Particular care needs to be taken when working with brackets that don't start with 1, and also when multiplying out all the terms to arrive at the final simplified series (up to and including the power required).

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Chapter 5 – Radians

Chapter 5 – Radians (Part A – Radians) By the end of the sub-unit, students should: • understand the definition of a radian and be able to convert between radians and degrees; • know and be able to use exact values of sin, cos and tan; • be able to derive and use the formulae for arc length and area of sector.	Ensure all students know how to change between radian and degree mode on their own calculators and emphasise the need to check which mode it is in. Radian measure will be new to students and it is important that they understand what 1 radian actually is. Make sure students know that 'exact value' implies an answer must be given in surd form or as a multiple of π. They need to know the exact values of sin and cos for 0, $\frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}, \pi$ (and their multiples) and exact values of tan for 0, $\frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}, \pi$ (and their multiples). Emphasise the need to always put a scale on both axes when drawing trigonometric graphs; students must be able to do this in radians. Make links between writing the trig ratio of any angle (obtuse/reflex/negative) to the trig ratio of an acute angle and to the trig graphs. (Do not rely on the CAST method as this tends to show a lack of understanding.) Derive the formulae for arc length and area of a sector by replacing the $\frac{\theta}{360°}$ in the GCSE formulae with $\frac{\theta}{2\pi}$. The πs cancel giving length of arc = $r\theta$ and area of sector = $\frac{1}{2}r^2\theta$. Cover examples which will involve finding the area of a segment by subtracting a triangle from a sector.
Chapter 5 – Radians (Part B – Small Angles) By the end of the sub-unit, students should: • understand and be able to use the standard small angle approximations for sine, cosine and tangent.	The Specification states:- $\sin\theta \approx \theta$, $\cos \approx 1 - \frac{\theta^2}{2}$, $\tan\theta \approx \theta$(where θ is in radians) Experiment with trigonometric graphs and a graph-drawing package by reading off values near the origin and zooming into small angles so the students get a feeling for this new concept. The formal proof is based on considering the area of a sector in which the angle is so small, the shape becomes a right-angled triangle (since the curved part is straightened). By considering the area of the triangle within the sector, the area of the sector and the area of the right angled triangle we can see that $\frac{1}{2}r^2\sin\theta < \frac{1}{2}r^2\theta < \frac{1}{2}r^2\tan\theta$ Cancelling $\frac{1}{2}r^2$ gives $\sin\theta < \theta < \tan\theta$ Dividing by $\sin\theta$ gives $1 < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta}$ As θ tends to 0, $\frac{1}{\cos\theta}$ tends to 1, and so $\frac{\theta}{\sin\theta}$ must tend to 1 as it is fixed between two values which tend to 1. So $\frac{\theta}{\sin\theta}$ is approximately equal to 1 for small values of θ (the small angle was the assumption at the start). Rearranging gives $\sin\theta \approx \theta$. Following a similar process, but dividing by $\tan\theta$ at the start gives $\tan\theta \approx \theta$. Using the identity $\cos\theta = 1 - 2\sin^2\frac{1}{2}\theta$ (which is covered in a later sub-unit), and substituting $\sin\frac{1}{2}\theta \approx \frac{1}{2}\theta$, gives the third approximation $\cos\theta \approx 1 - \frac{\theta^2}{2}$. The small angle approximations can be used to give estimated values of trigonometric expressions. For example, $\frac{\cos 3x - 1}{x\sin 4x}$ approximates to $-\frac{9}{8}$ (when x is small)

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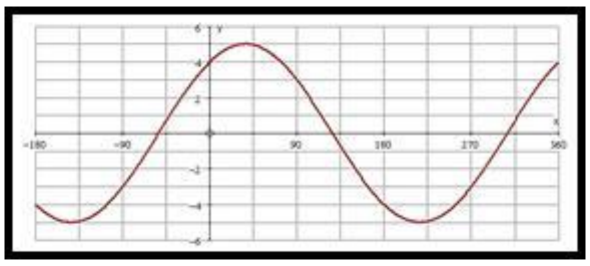
Chapter 6 – Trigonometric Functions

Chapter 6 – Trigonometric Functions - Inverse Trigonometric Functions By the end of the sub-unit, students should: • understand the secant, cosecant and cotangent functions, and their relationships to sine, cosine and tangent; • be able to sketch the graphs of secant, cosecant and cotangent; • be able to simplify expressions and solve involving sec, cosec and cot; • be able to solve identities involving sec, cosec and cot; • know and be able to use the identities $1 + \tan^2 x = \sec^2 x$ and $1 + \cot^2 x = \mathrm{cosec}^2 x$ to prove other identities and solve equations in degrees and/or radians • be able to work with the inverse trig functions $\sin^{-1}$, $\cos^{-1}$ and $\tan^{-1}$; • be able to sketch the graphs of $\sin^{-1}$, $\cos^{-1}$ and $\tan^{-1}$.	Introduce students to the reciprocal trigonometric functions secant θ, cosecant θ and cotangent θ. A good way to introduce these as reciprocal trig functions is to start by asking whether there is another way of writing x^{-1}. This should lead to the answer $\frac{1}{x}$. If we try this with $\sin^{-1}\theta$ it is not the same meaning as $\frac{1}{\sin\theta}$, so we need to name a different function $\mathrm{cosec}\,\theta$. (Contrast this with inverse trig functions looked at later in this section) To help students remember which reciprocal function goes with sin, cos and tan, point out that the **third** letter of these new functions, gives the name of the trig function in the denominator, i.e. se**c** $\theta = \frac{1}{\cos\theta}$ co**s**ec $\theta = \frac{1}{\sin\theta}$ co**t** $\theta = \frac{1}{\tan\theta}$ You should also point out that cot θ can be written as the reciprocal of tan θ to give $\frac{\cos\theta}{\sin\theta}$. Students will be expected to know what the graphs of each of the reciprocal and inverse functions look like and their key features, including domains and ranges. The relationships between the graphs and their originals can be explored on graphical calculators or graphing Apps. Show students how to work out new trigonometric identities by dividing $\sin^2\theta + \cos^2\theta$ = 1 (from AS Mathematics - Pure Mathematics) by $\cos^2\theta$ or by $\sin^2\theta$ to give the two new identities: $1 + \tan^2\theta = \sec^2\theta$ and $1 + cot^2\theta = \mathrm{cosec}^2\theta$. This is a good alternative to simply remembering the identities and lessens the chance of mixing them up. It is a good idea to use the new identities to solve trigonometric equations (which are often quadratic look-a-likes) before proving identities. Sub-unit 6f covers proving identities when all the available formulae have been covered.

Chapter/Objectives	Teaching Points

Chapter 7 – Trigonometry & Modelling

Chapter 7 – Trigonometry & Modelling (Part A – Compound, Double and Half Angle Formulae) By the end of the sub-unit, students should: • be able to prove geometrically the following compound angle formulae for sin $(A \pm B)$, cos $(A \pm B)$ and tan $(A \pm B)$; • be able to use compound angle identities to rearrange expressions or prove other identities; • be able to use compound angle identities to rearrange equations into a different form and then solve; • be able to recall or work out double angle identities; • be able to use double angle identities to rearrange expressions or prove other identities; • be able to use double angle identities to rearrange equations into a different form and then solve.	A good introduction is to ask the class to work out sin (30 + 60)°.It is equal to sin (90)° = 1. Go on to ask whether sin 30° + sin 60° gives the same value (either using a calculator or using surds). They should discover that the values are different. Explain that the reason for this is that you can't simply multiply out functions in this way. This leads in to explaining why compound angle formulae are needed to calculate sin $(A + B)$. Unit 1 above gives an example of a geometric proof by deduction for sin $(A + B)$. Care needs to be taken when using the result to extend to sin $(A - B)$ for negative values. Students will need to remember that cos $(-B)$ = cos B and that sin $(-B)$ = -sin (B). Extend these formulae by substituting A = B to derive the double angle formulae Show that there is only one version of $\sin 2x = 2\sin x \cos x$, but the basic version of $\cos 2x = \cos^2 x - \sin^2 x$, can be re-written by substituting $\cos^2 x + \sin^2 x = 1$ (from AS Mathematics - Pure Mathematics) into two different versions (exclusively in $\sin x$ or $\cos x$). ext sub- unit will look at how to solve equation A critical part of future questions and proofs involves choosing the correct version of the compound and/or double angle formulae.
Chapter 7 – Trigonometry & Modelling (Part B – R cos (x ± α) or R sin (x ± α)) By the end of the sub-unit, students should: • be able to express $\alpha \cos\theta + b\sin\theta$ as a single sine or cosine function; • be able to solve equations of the form $\alpha\cos\theta + b\sin\theta = c$ in a given interval.	Start by drawing a graph of, say, $4\cos x + 3\sin x$ to show that is has the basic sin-cos shape. Where are the coordinates of the maximum or minimum points? It approximately fits $5\cos(x - 40°)$.  Equating $4\cos x + 3\sin x$ to an expanded form of $R\cos(x - \alpha)$ gives: $$4\cos x + 3\sin x \equiv R\cos x\cos\alpha + R\sin x\sin\alpha$$ Equating coefficients leads to: $R\sin\alpha = 3$ and $R\cos\alpha = 4$. By squaring and adding we obtain $R = 5$, and by dividing we obtain $\alpha = 36.9°$. (This confirms the approximate fit above.) Move on to solving equations of the type $a\cos\theta + b\sin\theta = c$ using $R\cos(x \pm \alpha)$ or $R\sin(x \pm \alpha)$ as the first step. Effectively, the question reduces to a trigonometry equation like those done in Pure Paper 1, but at this level the angles could be in radians.

Chapter/Objectives	Teaching Points
Chapter 7 – Trigonometry & Modelling (Part C – Proving Trigonometric Identities) By the end of the sub-unit, students should: • be able to construct proofs involving trigonometric functions and previously learnt identities.	Proving trigonometric identities is something that challenges many students and is considered by some to be the most challenging part of the course. The basic principles are the same as in Unit 1 (Proof): manipulate the LHS and use logical steps to make it to match the RHS or vice-versa. (Sometimes both sides can be manipulated to reach the same expression.) Make sure you explain why we use ≡ rather than =. In the example below, the most efficient method is to start with the LHS and use $\sec^2\theta = 1 + \tan^2\theta$ to replace the numerator. The vital step is to multiply top and bottom of the resulting fraction by $\cos^2\theta$, this leads to the two familiar identities involving $\sin^2\theta$ and $\cos^2\theta$. (a) Prove that $\frac{\sec^2\theta}{1-\tan^2\theta} \equiv \sec 2\theta, \quad \theta \neq \frac{n\pi}{4}, n \in Z$ (b) Hence state a reason why the equation $\frac{\sec^2\theta}{1-\tan^2\theta} = \frac{1}{2}$ does not have any solutions. The final step has 'Hence', so students should be encouraged to use the result in part (a) and write $\sec 2\theta = \frac{1}{2}$, which leads to $\cos 2\theta = 2$. Students now need to explain fully that $-1 \leq \cos 2\theta \leq 1$, and so $\cos 2\theta = 2$ has no solutions.
Chapter 7 – Trigonometry & Modelling (Part D – Solving Problems in Context) By the end of the sub-unit, students should: • be able to use trigonometric functions to solve problems in context, including problems involving vectors, kinematics and forces.	Links can be made with simple harmonic motion in further mechanics, where a sin and/or cos curve could model the height of the tide against a harbour wall. When is it safe for the ship to come into the port? For kinematics the velocity equation could be expressed as $v = 3\sin(2t) ms^{-1}$. The times at which the object is stationary or at maximum speed could be analysed (no calculus at this stage). An oscillating share price could be modelled using trigonometric equations. Ask students: when is the best time to buy and sell?

Chapter/Objectives	Teaching Points

Chapter 8 – Parametric Equations

Chapter 8 – Parametric Equations (Part A – Converting between Parametric & Cartesian Forms) By the end of the sub-unit, students should: • understand the difference between the Cartesian and parametric system of expressing coordinates; • be able to convert between parametric and Cartesian forms.	Begin by explaining the difference between the Cartesian system, when a graph is described using $y = \mathrm{f}(x)$, and the parametric system, which uses $x = \mathrm{f}(t)$ and $y = \mathrm{g}(t)$ for some parameter t. Illustrate this by asking the class to consider $x = 5t$ and $y = 3t^2$ and to try to eliminate t from the two equations. This will give $y = \frac{3}{25}x^2$ or $25y = 3x^2$. (This is a quadratic equation - parabola.) Repeat for $x = 5t$ and $y = \frac{5}{t}$. This becomes $y = \frac{25}{x}$ (a hyperbola). Sometimes we need to eliminate the parameter, t, by using identities rather than substitution. Consider $x = 3\cos t$ and $y = 3\sin t$. Squaring both equations and adding means we can use $\cos^2 t + \sin^2 t = 1$ to give $x^2 + y^2 = 9$. (This is a circle, centre (0, 0) of radius 3.) Ask students to use similar methods to show that $x = 2 + 5\cos t, y = -4 + 5\sin t$ describes a circle centre (2, −4) with radius 5. How do we convert from Cartesian to parametric? (We need to be in radians) For example, what are the pair of parametric equations for a circle, centre (3, 5) radius 10?
Chapter 8 – Parametric Equations (Part B – Curve Sketching & Modelling) By the end of the sub-unit, students should: • be able to plot and sketch curves given in parametric form; • recognise some standard curves in parametric form and how they can be used for modelling.	It is often easier to match the properties of a curve in parametric form than it is in its Cartesian form. In order to establish the shapes of some well-known curves such as circles, ellipses etc., ask the students to plot the pair of parametric equations in the form of a table of values. When plotting $x = 4\cos t, y = 4\sin t$ what will the range of t be? (Remember to use radians.) Now plot $x = 4\cos t, y = 2\sin t$. (This is the shape mentioned in the reasoning/problem solving section of sub-unit 7a.) What values of t will we need for $x = 5t, y = \frac{5}{t}$? Investigate parametric equations which give closed loops. These will be integrated later in course to find the area of a loop, so we need to establish how values of t link plotting (direction vital). The specification states 'Students should pay particular attention to the domain of the parameter t, as a specific section of a curve may be described.'

Chapter/Objectives	Teaching Points

Chapter 9 – Differentiation

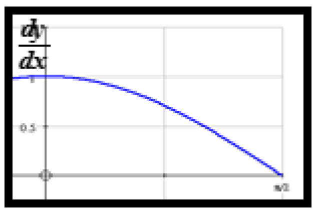
Chapter 9 – Differentiation (Part A – Differentiating Sin and Cos)

By the end of the sub-unit, students should:

- be able to find the derivative of $\sin x$ and $\cos x$ from first principles.

Review how to differentiate polynomials from first principles.

Sketch $y = \sin x$ and consider the gradient at key points by looking at slopes of tangents. If we plot the gradients then we get a shape which looks like the start of a cos graph:



This suggests that if $y = \sin x$, then $\frac{dy}{dx} = \cos x$, but this is not a proof or derivation!

Approach the differentiation from first principles in the same way as in AS Mathematics - Pure Mathematics, see SoW Unit 6.

Let's take a chord for $y = \sin x$ at $(x, \sin x)$ and $(x + \delta x, \sin(x + x))$, the gradient of the chord is

$\frac{\sin(x+\delta x)-\sin x}{\delta x}$

Using compound angle identity for $\sin(A + B)$ we find that $\frac{\sin x \cos \delta x + \cos x \sin \delta x - \sin x}{\delta x}$

By manipulation we obtain $\frac{\sin x(\cos \delta x - 1)}{\delta x} + \cos x \frac{\sin \delta x}{\delta x}$

Since $\delta x \to 0$, $\frac{\sin \delta x}{\delta x} \to 1$ and $\frac{\cos \delta x - 1}{\delta x} \to 0$ we conclude that

$$\lim_{\delta x \to 0} \frac{\sin(x + \delta x) - \sin x}{\delta x} = \cos x$$

Therefore the gradient of the chord → gradient of the curve and we conclude that $\frac{dy}{dx} = \cos x$.

A similar argument with $y = \cos x$ as a starting point leads to:

$$\frac{\cos(x + \delta x) - \cos x}{\delta x} = \frac{\cos x \cos \delta x - \sin x \sin \delta x - \cos x}{\delta x}$$

and therefore finding the derivative to be $-\sin x$.

The alternative notations $h \to 0$ rather than $\delta x \to 0$ are acceptable.

Chapter/Objectives	Teaching Points

<u>Chapter 9 – Differentiation (Part B – Differentiating Exponentials and Logarithms)</u>

By the end of the sub-unit, students should:

- be able to differentiate functions involving e^x, $\ln x$ and related functions such as $6e^{4x}$and $5\ln 3x$ and sketch the graphs of these functions;
- be able to differentiate to find equations of tangents and normals to the curve.

It is vital that students understand the functions e^x and $\ln x$ and do not just learn how to differentiate them. Use a graphing tool to show that the gradient of a special curve $y = a^x$ has a gradient which is exactly a^x. In other words its rate of growth is exactly the same as its value at that point. This models biological growth in nature (and decay if we consider a^{-x}) The curve sits between 2^x and 3^x and has a value of 2.718… We call this exponential e.

Therefore if $y = e^x$, $\frac{dy}{dx} = e^x$.

Explain that if $y = 2e^x$ then $\frac{dy}{dx} = 2e^x$.

The students could verify this on the graphs below as Fig. 1 is effectively a stretch parallel to the y-axis.

Fig. 2 shows that the graph of $y = e^{2x}$ is twice as steep as e^x, hence if $y = e^{2x}$ then $\frac{dy}{dx} = 2e^2x$.

These results will be deduced more formally in Unit 8c.

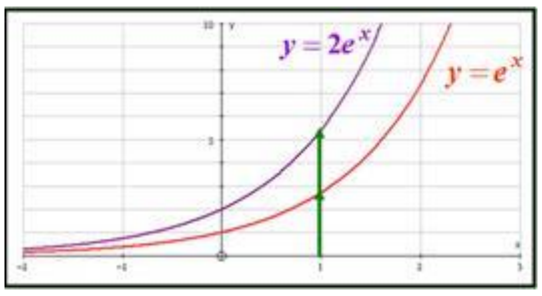


Figure 1

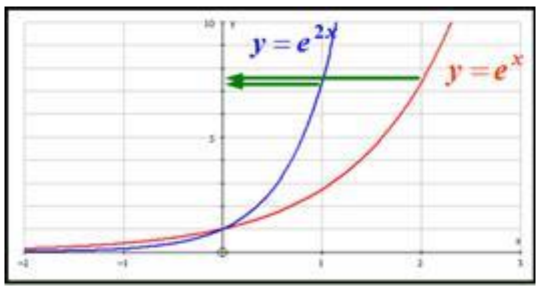


Figure 2

For natural logarithms, recap the basic definition and graphs (from Pure Paper 1)

By looking at the graph we can see that the gradient of $y = \ln x$ at any particular point is the reciprocal of the x-coordinate of that point where the tangent is drawn. Therefore for $y = \ln x$, $\frac{dy}{dx} = \frac{1}{x}$.

This can be derived in the following way:

If $y = \ln x$, then, from our definition of logs, $x = e^y$. [Write 2 = $\log_{10}100$ and 100 = 10^2 to illustrate this.]

We can differentiate $x = e^y$ by finding $\frac{dx}{dy}$ instead of the usual $\frac{dy}{dx}$.

$\frac{dx}{dy} = e^y$, and taking the reciprocal of both sides gives $\frac{dy}{dx} = \frac{1}{e^y}$.

We know that $e^y = x$ from above, so this gives $\frac{dy}{dx} = \frac{1}{x}$ as the derivative of $y = \ln x$.

The graphical approach could then be used to investigate why, for example, $y = \ln(3x)$ also has a derivative of $\frac{1}{x}$

Chapter/Objectives	Teaching Points

Chapter 9 – Differentiation (Part C – Differentiating Products, Quotients, Implicit and Parametric)

By the end of the sub-unit, students should:

- be able to differentiate composite functions using the chain rule;
- be able to differentiate using the product rule;
- be able to differentiate using the quotient rule;
- be able to differentiate parametric equations;
- be able to find the gradient at a given point from parametric equations;
- be able to find the equation of a tangent or normal (parametric);
- be able to use implicit differentiation to differentiate an equation involving two variables;
- be able to find the gradient of a curve using implicit differentiation;
- be able to verify a given point is stationary (implicit).

Most students will be able to differentiate simple instances of e^{3x}, $\sin 3x$ and $\ln 3x$ without needing formal methods such as $\frac{\mathrm{d}}{\mathrm{d}x}\ln \mathrm{f}(x) = \frac{\mathrm{f}'(x)}{\mathrm{f}(x)}$.

Many will also be able to differentiate expressions such as $(3x+7)^5$ without using the formal method $\frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{f}(x))^n = n(\mathrm{f}(x)^{n-1})\mathrm{f}'(x)$.

When using the chain rule and the formula $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x}$, initially u can be given to students, but they must be able to choose their own u and should move onto this quickly. Encourage students to lay work out carefully, using correct notation and $\frac{\mathrm{d}y}{\mathrm{d}u}$ and $\frac{\mathrm{d}u}{\mathrm{d}x}$, not always $\frac{dx}{dy}$.

Teaching should focus on *how* students know a function needs to be differentiated using the chain rule (or function of a function) and *why* a particular *u* is selected.

As an introduction for the product rule, ask the students to differentiate x^4. If you rewrite this as the product $(x^2)(x^2)$ and differentiate each part separately, it does not match $4x^3$. Using the product rule will give that match.

In a similar way, writing x^4 as $\frac{x^3}{x}$ can lead into the quotient rule.

Work involving the product and quotient rule often breaks down because of weak algebraic skills and this needs plenty of practice. Students should practice fully simplifying their answers as they may be asked to give a solution in a particular form. Encourage students to lay work out carefully. Good notation is vital to achieve success.

Show that the product rule and the quotient rule give the same answers on functions that can be written in two ways, for example, $y = \frac{x+1}{x+2}$ and $y = (x+1)(x+2)^{-1}$.

Also show that the chain rule and the product rule give the same derivative for $\cos^2 x$ and $\sin^2 x$.

Use the product and quotient rules to derive the differentials of some key trigonometric expressions. For example $\frac{\mathrm{d}}{\mathrm{d}x}(\tan x) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\sin x}{\cos x}\right)$ using the quotient rule giving $\sec^2 x$.

For parametric differentiation, make links with the chain rule to give $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t} \div \frac{\mathrm{d}x}{\mathrm{d}t}$

Stress that we often substitute in the value of the parameter t at the point which we need to find the gradient

Many questions will involve trigonometric functions, so students must be fluent at differentiating these.

For implicit differentiation, consider the equation of a circle, $x^2 + y^2 = 16$. To differentiate this function we would have to make y the subject of the formula. Sometimes this can be difficult or even impossible.

Make sure students can confidently differentiate terms like $x^2 y$ using implicit differentiation. Finally, stress that we need to substitute in *both* x and y coordinates to find the gradient at a certain point.

Students may have to apply the product or quotient rules in implicit differentiation questions and should be given examples of this. In exam questions students are almost always required to find the gradient through implicit differentiation.

Take a point on a circle or another type of curve and find the gradient using two both parametric and implicit differentiation. Then find the equation of tangent and/or normal and see that both methods give the same answer.

Chapter 9 – Differentiation (Part D – Second Derivatives: Rates of Change of Gradient, Inflections)

By the end of the sub-unit, students should:

- be able to find and identify the nature of stationary points and understand rates of change of gradient.

The specification states 'Understand and use the second derivative as the rate of change of gradient; connection to convex and concave sections of curves and points of inflection' and 'know that at an inflection point $\mathrm{f}''(x)$ changes sign.'

The basic principle is usually

	$\frac{\mathrm{d}y}{\mathrm{d}x}$ or $\mathrm{f}'(x)$	$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ or $\mathrm{f}''(x)$
maximum	$= 0$	< 0
minimum	$= 0$	> 0

However show examples of curves in which $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ or $\mathrm{f}''(x) = 0$, where there could beva point of inflexion (or not). i.e. The rate of change of gradient is zero.

We would need to work out $\mathrm{f}'(x)$ and scrutinise gradient either side of the point x. There may be positive or negative inflexion or neither (depending on the nature of the curve, which could be convex or concave).

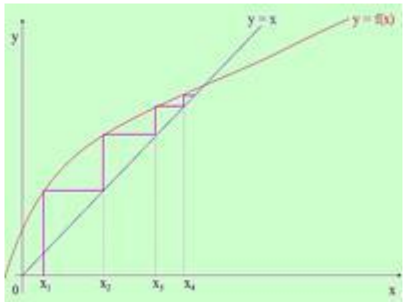
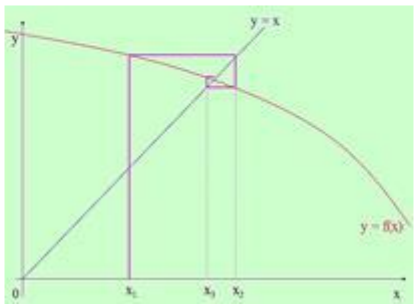
Use graph drawing packages to investigate the shapes and turning points of various curves of the type $y = ax^n (n > 2)$

Chapter/Objectives	Teaching Points

Chapter 9 – Differentiation (Part E – Rates of Change Problems: Differential Equations) By the end of the sub-unit, students should: • be able to use a model to find the value after a given time; • be able to set up and use logarithms to solve an equation for an exponential growth or decay problem; • be able to use logarithms to find the base of an exponential; • know how to model the growth or decay of 2D and 3D objects using connected rates of change; • be able to set up a differential equation using given information which may include direct proportion.	This content links to kinematics, where velocity is considered as $\frac{\mathrm{d}s}{\mathrm{d}t}$ and acceleration as $\frac{\mathrm{d}v}{\mathrm{d}t}$. The example below is from the original SAMs: A team of conservationists is studying the population of meerkats on a nature reserve. The population is modelled by the differential equation $\frac{\mathrm{d}P}{\mathrm{d}t} = \frac{1}{22}P(11-2P),\ t \geq 0$ where P, in thousands, is the population of meerkats and t is the time measured in years since the study began. Given that there are 1000 meerkats on the nature reserve when the study began, (a) determine the time taken, in years, for this population of meerkats to double, (b) show that the population cannot exceed 5500.

Chapter/Objectives	Teaching Points

Chapter 10 – Numerical Methods

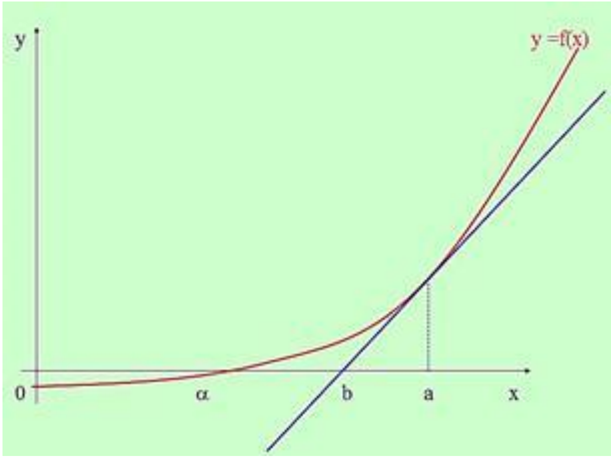
Chapter 10 – Numerical Methods (Part A – Location of Roots) By the end of the sub-unit, students should: • be able to locate roots of $\mathrm{f}(x) = 0$ by considering changes of sign of $\mathrm{f}(x)$; • be able to use numerical methods to find solutions of equations.	Students should be able to recognise that a root exists when there is a change of sign of $\mathrm{f}(x)$. Students should recognise this and remember it. There is often an easy mark missed on the exam for this because it is phrased slightly differently. Students should know that sign change is appropriate for continuous functions in a small interval. When the interval is too large the sign may not change as there may be an even number of roots. If the function is not continuous, the sign may change but there may be an asymptote (not a root) so the method will fail.
Chapter 10 – Numerical Methods (Part B – Solving by Iterative Methods 'Staircase & Cobweb') By the end of the sub-unit, students should: • understand the principle of iteration; • appreciate the need for convergence in iteration; • be able to use iteration to find terms in a sequence; • be able to sketch cobweb and staircase diagrams; • be able to use cobweb and staircase diagrams to demonstrate convergence or divergence for equations of the form $x = \mathrm{g}(x)$.	Students will have met iterations at GCSE (9-1) Mathematics, but will need to be introduced to some of the conditions for convergence and understand how the process works (and sometimes does not work). Revise the method to make one of the x 's the subject of the formula, leading to $x = \mathrm{f}(x)$. Use graph-drawing packages to look at the function and decide where would be appropriate for the first iteration value (i.e. x_0). The method at A level is to consider the roots of the function $y = \mathrm{f}(x)$ as the intersection of the two functions $y = x$ and $y = \mathrm{f}(x)$ (hence $x = \mathrm{f}(x)$). Use an iteration of the form $x_{n+1} = \mathrm{f}(x_n)$ to find a root of the equation $x = \mathrm{f}(x)$ and show how the convergence can be understood in geometrical terms by drawing cobweb and staircase diagrams like those shown here.  

Chapter/Objectives	Teaching Points

Chapter 10 – Numerical Methods (Part C – Newton-Raphson Method)

By the end of the sub-unit, students should:

- be able to solve equations approximately using the Newton-Raphson method;
- understand how the Newton-Raphson method works in geometrical terms.

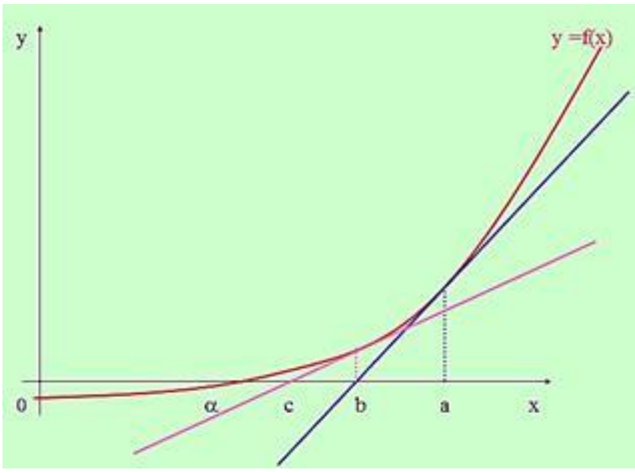


Consider the diagram above. The tangent crosses the x-axis at b (which is quite near the actual root α).

By considering the gradient of the tangent, we get $f'(a) = \frac{f(a)}{a-b}$ which can be rearranged to give $b = a - \frac{f(a)}{f'(a)}$.

We therefore have an expression for an approximation of the root (b, which uses the equation of the curve and its derivative at the point a.

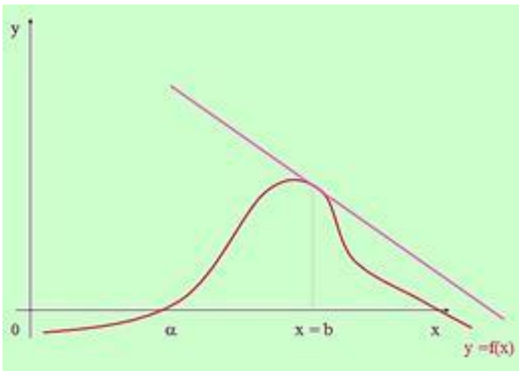
If we now go up from the point b, hit the curve and then construct another tangent (as in the diagram below) then, a similar argument, gives a better approximate root at c (nearer than b). Therefore we would get $c = b - \frac{f(a)}{f'(a)}$.



So if we continued this process we would get $d = c - \frac{f(a)}{f'(a)}$ and generally $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$.

Sometimes the process fails for some curves or starting points.

What happens to the tangent if we try to apply the process here?

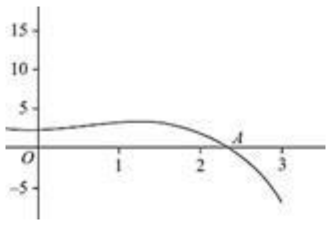


An example of the type question which may be seen:

$f(x) = x^3 + 8x - 19$.

Obtain an approximation to the real root of $f(x) = 0$ by performing two applications of the Newton-Raphson procedure to $f(x)$, using $x = 2$ as the first approximation.

Give your answer to 3 decimal places.

Chapter/Objectives	Teaching Points
Chapter 10 – Numerical Methods (Part D – Problem Solving) By the end of the sub-unit, students should: • be able to use numerical methods to solve problems in context.	Recurrence relations, iterations and Newton-Raphson methods can be used obtain approximate solution(s) to an equation set in a context. The important point to make is that the original equation is too difficult to solve algebraically (e.g. the roots are decimal and/or the functions will not factorise or contain terms which are non-polynomials). The choice of degree of accuracy is dependent upon the context of the problem, e.g. nearest minute or number of years. An example of a possible question is as follows. The equation $P = -t^3 + 2t^2 + 2 (t > 0)$ represents a share price p, at time t months after the money was invested. The iteration $t_{n+1} = \frac{2}{(t_n)^2} + 2$ represents the solution to the above equation.  Taking t_0 = 2.5 months, show that the root gives an approximation to when the share price has zero value. Use the iteration to find the (converged) time at which the shares lose their value before going negative. When were the shares at their highest value? Can Newton-Raphson be used to find the approximate solution of the above relationship?

Chapter/Objectives	Teaching Points

Chapter 11 – Integration

Chapter 11 – Integration (Part A – Exponentials, Trigonometric & Parametric Functions) By the end of the sub-unit, students should: • be able to integrate expressions by inspection using the reverse of differentiation; • be able to integrate x^n for all values of n and understand that the integral of $\frac{1}{x}$ is $\ln \|x\|$; • be able to integrate expressions by inspection using the reverse of the chain rule (or function of a function); • be able to integrate trigonometric expressions; • be able to integrate expressions involving e^x; • be able to integrate a function expressed parametrically;	Recap all the methods of differentiation covered earlier in the course. This can also be used as a starting point for introducing the different rules for integration. Consider the integral of $x^{-1} = \frac{1}{x}$. Using the rule from AS Mathematics - Pure Mathematics gives. $\frac{1}{0}$. However, if we recall that the differential of $\ln \|x\|$ is $\frac{1}{x}$, then the reverse operation tells us that the integral of $\frac{1}{x}$ is $\ln \|x\| + c$. Similarly, the differential of e^x is e^x, so the integral will also give the same result. Finally, the differential of trig expressions should be recapped as this also leads to some standard results for trigonometric integrals. Take care to show how the integral of $\sin x$ is $-\cos x + c$ (as the differential of $\cos x$ leads to $-\sin x$). The integral of $\sec^2 x$ looks difficult but is only the reverse of the differential of $\tan x$. Students must end all indefinite integrations with $+c$ and use correct notation when integrating and must include $\mathrm{d}x$. Encourage students to develop their own technique for integrating problems which require the reverse chain rule. If good examples are used, most students will be able to work out their own method and soon be able to write down the answers directly for integrals like $3e^{2x}$ and $4 \sin(3x)$. End this section by explaining how the formula $\int y \left(\frac{dx}{dt}\right) dt$ can be used to integrate a pair of equations expressed parametrically.
Chapter 11 – Integration (Part B – Reverse of Differentiation) By the end of the sub-unit, students should: • recognise integrals of the form $\int \frac{f'(x)}{f(x)} \mathrm{d}x = \ln \|f(x)\| + c$; • be able to use trigonometric identities to manipulate and simplify expressions to a form which can be integrated directly.	Consider the rule for differentiating $\ln \|f(x)\|$. This was $\frac{f'(x)}{f(x)}$. A special case of this is the integral of $\frac{1}{x}$, which is $\ln \|x\| (+c)$. So, if we have to integrate an expression in which the top of the fraction is the exact differential of the denominator (or a multiple of it), then the answer is the natural log of the denominator $(+c)$. Make sure students can adjust questions like the integral of $\frac{4x^2}{x^3}$. Consider examples like the integral of $\tan x$ by rewriting it as $\frac{\sin x}{\cos x}$, leading to a natural log answer (be careful with the minus!) One of the most common integrals is $\cos^2 x$. The standard method for integrating this is to rearrange the appropriate double angle formula to create an integral involving not x^2 but $2x$ which is much easier to directly integrate (as shown in the previous section). Students will need lots of practice in selecting the correct version of $\cos 2x$, which involves only $\cos^2 x$ terms and then rearranging it. The specification states: 'students are expected to be able to use trigonometric identities to integrate, for example, $\sin^2 x$, $\tan^2 x$, $\cos^2 3x$ '.

Chapter/Objectives	Teaching Points

Chapter 11 – Integration (Part C – By Substitution) By the end of the sub-unit, students should: • be able to integrate expressions using an appropriate substitution; • be able to select the correct substitution and justify their choices.	Most students find integration by substitution challenging and will need to complete lots of different styles of questions. It is a good idea to start with an example which can be performed by inspection as the reverse of differentiation. Students also like to have a step by step process. 3. Use the given substitution or decide on your own. The substitution is usually the contents of a bracket, square root or the 'nasty' bit! i.e. Let u = … 4. Differentiate the substitution i.e. $\frac{\mathrm{d}u}{\mathrm{d}x}$ = … 5. Make $\mathrm{d}x$ the subject of the formula 6. Replace the $\mathrm{d}x$ and make the substitution into the integrand 7. Cancel out any remaining x* 8. Integrate the resulting (simpler) integral 9. Substitute back to get the answer in terms of x again *If there are any remaining x, you can re-use the substitution making the x the subject For expressions including trigonometric functions, the identities involving $\sin^2 x$, $\sec^2 x$ are often useful to simplify the integrand.
Chapter 11 – Integration (Part D – By Parts) By the end of the sub-unit, students should: • be able to integrate an expression using integration by parts; • be able to select the correct method for integration and justify their choices.	It is a good idea to show how the product rule for differentiation can be integrated on both sides to derive the 'by parts' formula (which is given in the formulae booklet). Students are usually able to start questions using this method but struggle to get to full solutions and will require lots of practice with algebraic manipulation. Time should be spent discussing the choice of u and $\mathrm{d}v$. It is usually advisable to select the polynomial to be the u as it simplifies to a lower power after calculating $\mathrm{d}u$, thus making the second integral easier than the original question. Students should recognise that $\ln x$ cannot be integrated simply and should therefore always be chosen as u. $\ln x$ itself can be integrated using this method taking $u = \ln x$ and $\mathrm{d}v = 1$ (as we cannot integrate $\ln x$, but can differentiate it to give $\frac{1}{x}$). The $\mathrm{d}v$ becomes more complicated, but then simplifies in the second integral with the $\frac{1}{x}$. More able students should be able to access questions where it is necessary to use integration by parts twice (e.g. $u = x^2$).
Chapter 11 – Integration (Part E – By Partial Fractions) By the end of the sub-unit, students should: • be able to integrate rational expressions by using partial fractions that are linear in the denominator; • be able to simplify the expression using laws of logarithms.	Revise the simplification of rational expressions into partial fractions. We have already seen that this technique is useful in binomial expansions. Often the first part of an integration question of this sort will ask students to split the fraction into two (or more) partial fractions. The next part will then ask for the integration to be carried out. For example: Integrate $\int \frac{5}{(x-1)(3x+2)}\,\mathrm{d}x$. This will lead to $\int \frac{5}{(x-1)(3x+2)}\,\mathrm{d}x = \int \left(\frac{1}{x-1} - \frac{3}{3x+2}\right)\mathrm{d}x = \ln(x-1) - \ln(3x+2) (+c)$ It is sometimes sufficient to leave the answer in this form, but 'Show that' questions will influence the further simplification using laws of logs.

Chapter/Objectives	Teaching Points

Chapter 11 – Integration (Part F – Area Under graph)

By the end of the sub-unit, students should:

- understand and be able to use integration as the limit of a sum;
- understand the difference between an indefinite and definite integral and why we do not need + c;
- be able to integrate polynomials and other functions to find definite integrals, and use these to find the areas of regions bounded by curves and/or lines;
- be able to use a definite integral to find the area under a curve and the area between two curves.
- be able to find an area under a curve defined by a pair of parametric equations.

Begin by showing a sketch of the curve and spit the area below it into thin strips, as shown below.

Now each strip is of elemental width δx, so the approximate area of each strip is $y\delta x$, where y is the height of each strip measured on the y-axis. If we sum all the strips, this would give us the total area below the curve. If the first strip starts at the point (2, 0) and the last strip ends at (4, 0), these become the limits on the definite integral. We can think of '4' as the area up to 4 and '2' as the area up to 2 (both measured across from the y-axis or x = 0).

We have seen from work on series, that we can use the sigma notation for sums so we can represent the area as $\sum y\delta x$. As δx gets thinner and thinner, the area becomes more accurate as the strips become more like rectangles. (This links nicely with the trapezium rule in the next sub-unit.)

We say that 'in the limit, as δx approaches zero' the sum becomes continuous rather than discrete and we can replace y with $\mathrm{f}(x)$ and $y\delta x$ becomes $\mathrm{f}(x)\delta x$.

It happens that the rule for integration (which so far has only been used as the reverse of differentiation) gives the exact area under the curve. We can substitute in a and b, where the area's strips began and ended, as the limits of integration. The $y\ \delta x$ becomes $\mathrm{f}(x)\delta x$ and for the integral becomes $\mathrm{f}(x)\mathrm{d}x$. In other words the δx is the $\mathrm{d}x$ we have always understood as 'with respect to x'.

This leads to,

$$\int_a^b \mathrm{f}(x)\mathrm{d}x = \lim_{\delta x \to 0} \sum_{x=a}^{b} \mathrm{f}(x)\delta x$$

Do lots of work on finding areas that require more than just a simple integral to be evaluated, for example when some of the area is below the x-axis or when finding the area between a line and a curve.

For example:

Find the finite area bounded by the curve $y = 6x - x^2$ and the line $y = 2x$.

Find the finite area bounded by the curve $y = x^2 - 5x + 6$ and the curve $y = 4 - x^2$.

Encourage students to always do a sketch or use a graph drawer to help with such questions.

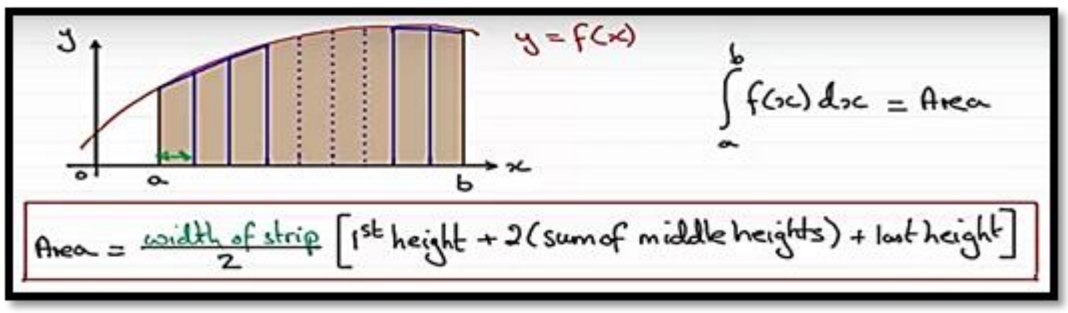
Chapter 11 – Integration (Part G – Trapezium Rule)

By the end of the sub-unit, students should:

- be able to use the trapezium rule to find an approximation to the area under a curve;
- appreciate the trapezium rule is an approximation and realise when it gives an overestimate or underestimate.

Make a direct link with the previous section and how to find an estimate for the area under a curve by dividing it into a finite number of strips. Sometimes an estimate is all that we need, and sometimes the integral is very complicated (or sometimes impossible) to integrate and so we have to estimate the area numerically.

The trapezium rule is given in the formula book (and may have also been covered in GCSE (9-1)). Students who struggle with algebra sometimes prefer to use the word version below:



Some students may be able to derive the rule by adding all the individual strips areas (i.e. $\frac{1}{2}h(y_0 + y_1) + \frac{1}{2}h(y_1 + y_2) + \ldots$) and then factorising to give the trapezium rule as in the formula book.

Ask students to calculate $\int_0^1 x\,\mathrm{e}^{2x}$ by using integration by parts and also by completing the table and using the trapezium rule (this is the quicker method). They should compare the answers they get using the different methods.

x	0	0.2	0.4	0.6	0.8	1
$y = xe^{2x}$	0	0.29836		1.99207		7.38906

Another example of the type of question that may be asked is:

Evaluate $\int_0^1 \sqrt{2x+1}\,\mathrm{d}x$ using the values of $\sqrt{2x+1}$ at x = 0, 0.25, 0.5, 0.75 and 1.

Make a sketch of the graph to determine whether the trapezium rule gives an over-estimate or an under-estimate of the exact value of the integral.

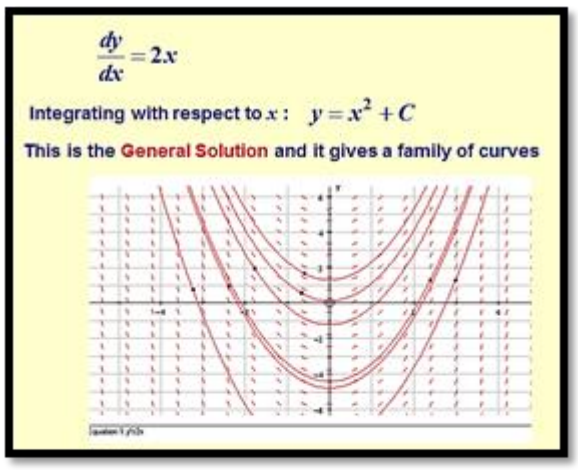
Chapter/Objectives	Teaching Points

Chapter 11 – Integration (Part H – Differential Equations)

By the end of the sub-unit, students should:

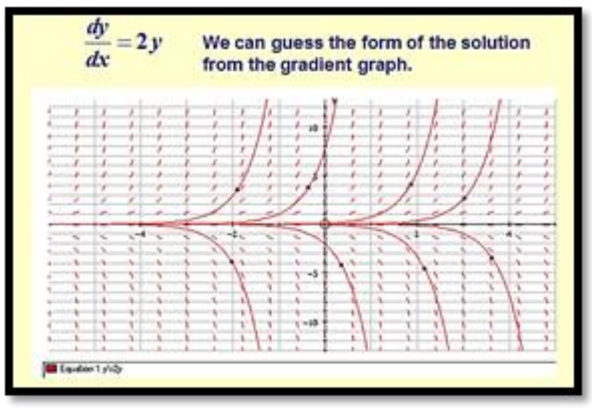
- be able to write a differential equation from a worded problem;
- be able to use a differential equation as a model to solve a problem;
- be able to solve a differential equation;
- be able to substitute the initial conditions or otherwise into the equation to find + c and the general solution.

Begin by considering the simplest possible differential equation (defined as first order) as below.



Notice that the graph drawing tool can plot the differential equation to give a family of curves which mirror the solution (family of parabolas)

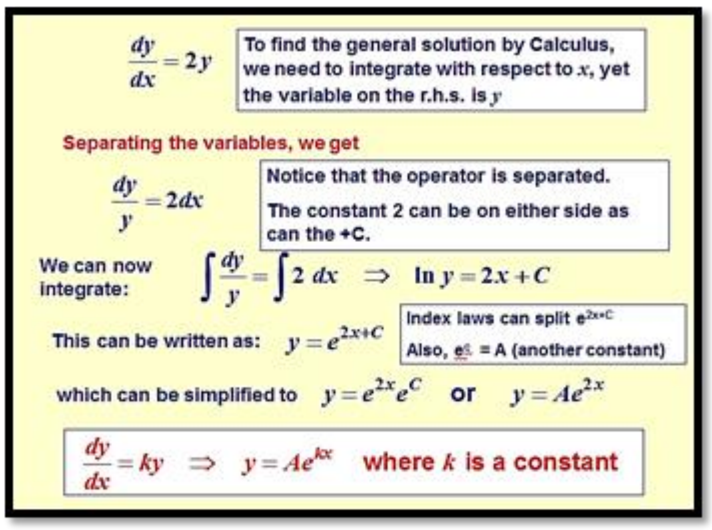
The next differential equation is more difficult as we cannot integrate directly because the variable is y rather than x. But looking at the family of curves may give us a clue about the eventual solution.



The curves look like exponentials.

The solution can be performed by using a method called 'separating variables', in which we rearrange and split up the $\frac{dy}{dx}$ as if it is a fraction. It is vital to keep all the y's and $\mathrm{d}y$'s *and* the x's and $\mathrm{d}x$'s together, but also the $\mathrm{d}x$ and $\mathrm{d}y$ must be in the numerator on each side.

The full solution is shown below.



As suspected, the family of curves were exponential curves.

$y = Ae^{2x}$is a general solution, but how do we find the value of the constant A? We need to have some information about the data from which the differential equation originates. Something along the lines of 'when x = 0, y = 2'.

Substituting this pair of values into the general solution and finding the value of A, will lead to a particular solution.

Sometimes we may have a choice of pairs to substitute or we may have two pairs of values in order to work out two constants.

Explain that questions may be set in a context and, in these cases, students need to interpret the solution of the differential equation in the context of the problem. This may including identifying limitations of the solution.

The following example is typical:-

The population of a town was 50 000 in 2010 and had increased to 55 000 by 2015. Assuming that the population is increasing at a rate proportional to its size at any time, estimate the population in 2020 giving your answer to the nearest hundred.

$\frac{db}{dt} = kn \Rightarrow n = Ae^{kt}$ as above, but now n is the number of people and t is the time in years.

Chapter/Objectives	Teaching Points
	The validity of the solution for large values should be considered, for example, if the question was modelling population growth; would it be realistic for the value to keep increasing forever?

Chapter 12 – Vectors

Chapter 12 – Vectors	
By the end of the sub-unit, students should: • be able to extend the work on vectors from AS Pure Mathematics to 3D with column vectors and with the use of **i**, **j** and **k** unit vectors; • be able to calculate the magnitude of a 3D vector; • know the definition of a unit vector in 3D; • be able to add 3D vectors diagrammatically and perform the algebraic operations of vector addition and multiplication by scalars, and understand their geometrical interpretations; • understand and use position vectors, and calculate the distance between two 3D points represented by position vectors; • be able to use vectors to solve problems in pure mathematics and in contexts (e.g. mechanics).	Begin by showing some 3D coordinates on x, y, z axes. (Graph drawing packages are very useful here, especially if you can turn the grid to view from different positions.) Consider a cuboid (2 by 3 by 4), with one corner at the origin. Ask the class to write down the coordinates of all the vertices. Remind students of 2D work and extend to 3D column vectors, orthogonal unit vectors **i**, **j**, **k** and position vectors. Write all the vectors from the 'origin' corner of the cuboid as position vectors (e.g. OA = **a**, etc.) Calculate the magnitude of these vectors as $\sqrt{2^2 + 3^2 + 4^2}$ for example. Extend this idea to calculating the distance d between two points (x_1, y_1, z_1) and (x_2, y_2, z_2) using $d^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2$ Extend the ideas of vector addition and subtraction to 3D: $\overrightarrow{OB} - \overrightarrow{OA} = \overrightarrow{AB}$ = **b** − **a**. Cover the triangle and parallelogram laws of addition, as well as demonstrating parallel vectors. Show how to find a unit vector in the direction of a, and make sure students are familiar with the notation | ***a*** | (extended to 3D). Use vectors to solve problems in pure mathematics and discuss the 3D geometrical interpretations of solutions.