Department of Mathematics
Year 13 Scheme of Work – Mechanics

3 – 2 = 1 → $\sin^2\theta + \cos^2\theta = 1$ → $\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{-\frac{1}{2}t^2}\,dt = 1$

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MOMENTS – FORCES' TURNING EFFECT

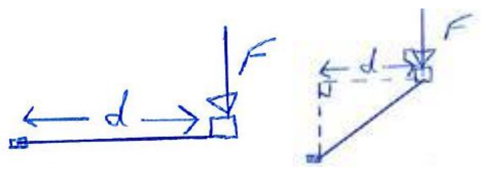
Chapter 4 – Moments Turning Effect

- By the end of the sub-unit, students should:
- realise that a force can produce a turning effect;
- know that a moment of a force is given by the formula force × distance giving Nm and know what the sense of a moment is;
- understand that the force and distance must be perpendicular to one another;
- be able to draw mathematical models to represent horizontal rod problems;
- realise what conditions are needed for a system to remain in equilibrium;
- be able to solve problems when a bar is on the point of tilting.

Start by asking two students to push up and down equally on two points of a ruler (or rod/beam) which are directly above or below each other. The forces balance and if we resolve vertically, the resultant force is zero. Hence the ruler will not move (equilibrium). However, if the two positions are separated, the ruler will turn, despite the forces still having no resultant in the vertical direction. So if two (or more) forces are not concurrent, there may be a turning effect. (See diagrams below.)

Next think about a door handle and imagine it was moved nearer the hinge of the door. Common sense tells us the door will be harder to open or close, so any formula for the turning effect of forces must involve distance as well as force.

Show a bicycle pedal in different positions and discuss which one makes turning easier. (See diagrams below.)



A discussion around this can lead to the understanding that the moment of a force, is a measure of its turning effect and is given by the formula:

moment of a force about a point = force (F) × perpendicular distance from the point to the line of action of the force (d) (the unit is newton metres, N m)

Ask students questions such as: How do we work out the distance, d, in the second bicycle pedal diagram? What additional information do we need? What if the pedal was at the topmost point, vertically above the axle?

The force and distance must be perpendicular to one another, but in this unit we will only be considering horizontal bars, supported or suspended by reactions and tensions respectively. These forces will naturally be vertical and parallel to one another, so the moments formula can be applied easily and the only thing to consider is the *sense* of the moment (whether the turning effect of each force is clockwise (negative) or anticlockwise (positive)).

Demonstrate that a uniform ruler will balance about its centre (where all the weight acts) and that this central point is therefore its centre of mass. Use this to extend students' basic idea of equilibrium as a system where there is no resultant force and also no overall turning effect, i.e. $R(\uparrow)$ = 0 N and the sum of the moments = 0 N m.

Make sure all the assumptions are revisited from earlier in the course e.g. model a rod as a straight line, a person standing on a bridge as a particle, strings being inextensible etc. The centre of mass is at the centre of the rod only if it is stated as being uniform.

Before starting on questions, make sure students know the notation: when we 'take moments' about a certain point (say A), we write this as M(A). Cover questions that involve:

rods resting on two or more supports

a rod which is suspended at two or more points

finding the position of the centre of mass of a non-uniform rod.

Make sure you stress that theoretically we can take moments about any point and, together with resolving (vertically), we can solve any problem. However, some positions will make the solution more efficient and subsequently involve less algebra. Show students that taking moments about a point through which a force acts is zero as the distance to that force is zero.

Chapter/Objectives	Teaching Points

FORCES AND FRICTION

Chapter 5 – Forces and Friction (Part A Resolving Forces) By the end of the sub-unit, students should: • understand the language relating to forces; • be able to identify the forces acting on a particle and represent them in a force diagram; • understand how to find the resultant force (magnitude and direction); • be able to find the resultant of several concurrent forces by vector addition; • be able to resolve a force into components and be able to select suitable directions for resolution.	Begin by considering two forces acting at right angles to one another (horizontal and vertical), use Pythagoras and trigonometry to find the hypotenuse (resultant R) and angle (direction θ above the horizontal) respectively. [You could also link to velocity from speed and vector addition rule.] It is easy going from component form to magnitude/direction; but can we go backwards? Guide students to consider the right-angled triangle and use trigonometry to show that the horizontal component is $R\cos\theta$ and the vertical component is $R\sin\theta$ of the Resultant, R (hypotenuse). Show that forces given in the form $\mathbf{i}$, $\mathbf{j}$ can be simply drawn as a right-angled triangle and the resultant and direction can be found the same way. Extend to finding the resultant of a system of forces given in $\mathbf{i} - \mathbf{j}$ form by adding **i** and **j** components. Look at two forces acting at *any* angle and show that the triangle can be solved using the cosine rule (to find the resultant) and sine rule (to find the direction). Extend to more than two forces and resolve the system using $R(\rightarrow)$ and $R(\uparrow)$ to create two perpendicular forces, then use Pythagoras and trigonometry to calculate the resultant and direction. Show that the weight component of a particle on an inclined plane acts in two directions: along and perpendicular to the plane. This will be a critical skill for solving the statics/dynamics questions in the next unit.
Chapter 5 – Forces and Friction (Part B Friction Forces) By the end of the sub-unit, students should: • understand that a rough plane will have an associated frictional force, which opposes relative motion (i.e. the direction of the frictional force is always opposite to how the object is moving or 'wants' to move); • understand that the 'roughness' of two surfaces is represented by a value called the coefficient of friction represented by μ; • know that 0 ≤ μ but that there is no theoretical upper limit for μ although for most surfaces it tends to be less than 1 and that a 'smooth' surface has a value of $\mu = 0$; • be able to draw force diagrams involving rough surfaces which include the frictional force • understand and be able to use the formula $F \leq \mu R$.	Start by asking students to rub their hands together vigorously. The warmth is caused by microscopic peaks and troughs on the surface of the skin interlocking. The rougher the surface, the 'sharper' these peaks and troughs. Explain to students that this principle applies even to the smoothest looking surfaces and the force which opposes motion is called the frictional force. The value which represents the roughness is called the coefficient of friction (μ) and is zero for a smooth surface. If we consider a book on a rough horizontal table, it will be *harder* to move the book if:- • we put a 'paperweight' on it (increasing the reaction force) or • we put it on a rougher surface (increasing the value of μ). Therefore the expression to model frictional forces uses these two factors (in direct proportion) and is given by μR. This is the maximum resistance any surface can provide before the book begins to move, so the inequality $F \leq \mu R$ applies until the force wanting to cause motion reaches the limiting value μR, called limiting friction. Consider a 10 kg book on a rough horizontal plane. If μ = 0.5, investigate the value of the frictional force if the pushing force, P is **a** 10 N, **b** 98 N, **c** 100 N [Link to *resultant force = ma* from AS Mathematics - Mechanics content, see SoW Unit 8.] Now place the book on an inclined plane and analyse the limiting friction being careful to stress that the reaction force is NOT the weight in this case. Will the book begin to slide for different angles of plane? What is the maximum angle achievable before the book slides?

Chapter/Objectives	Teaching Points

APPLICATIONS OF KINEMATICS

Chapter 6 – Applications of Kinematics (Projectiles) By the end of the sub-unit, students should: • be able to find the time of flight of a projectile; • be able to find the range and maximum height of a projectile; • be able to derive formulae to find the greatest height, the time of flight and the horizontal range (for a full trajectory); • know how to modify projectile equations to take account of the height of release; • be able to derive and use the equation of the path.	Define a projectile as an object dropped or thrown in the air. Show a video from the net of a golf chip or shot-putter. Explain that the path is called a parabola which is the old Greek word for throw. Discuss the modelling assumptions: the object is treated as a particle so it does not spin and has no air resistance. Therefore the only force on the object is gravity. (Link this back to vertical motion under gravity.) Discuss the fact that displacement, velocity and acceleration are vectors with components in the horizontal and vertical directions. These components obey the *suvat* formulae, and the horizontal and vertical directions can be treated separately. Begin with *horizontal* projection examples and encourage student to make two lists for the motion in the horizontal and vertical directions. It is easier to start this way as the initial *vertical* velocity is zero for this type of question, hence u = 0 for the vertical equation of motion. For **all** Projectile questions: a = 0 (in the horizontal direction), so the horizontal velocity is constant. a = 9.8 m s^{-2} or − 9.8 m s^{-2} (in the vertical direction) depending on whether downwards is taken as positive or upwards is taken as positive. The two equations of motion often will have time, t, as a common term. Move on to projection with speed U m s^{-1} at *any* angle α (above the horizontal ground) and introduce the concept of the initial velocity having horizontal and vertical components. (It may be advisable to revise magnitude and direction, Pythagoras and basic trigonometry.) Horizontally, $u = U\cos\alpha$ and if upwards is positive, vertically, $u = U\sin\alpha$. Derive the formulae for the time of flight, greatest height (when the vertical velocity is zero) and horizontal range (for the maximum range you will need to use the identity $\sin 2\alpha = 2\sin\alpha\cos\alpha$ which is covered in A level Mathematics - Pure Mathematics content, see SoW Unit 6d) Emphasise the fact that s is displacement. So, for example, for the vertical equation of motion, we use s = 0 if the projectile returns to the ground, and if it is projected from a height and lands lower than its starting point, then, if upwards is positive s will be negative in the vertical direction. Show examples with the initial velocity as an **i - j** vector (the **i** coefficient is u for the horizontal equation of motion). This actually makes it easier as the components are done for you.

Chapter/Objectives	Teaching Points

APPLICATIONS OF FORCES

Chapter 7 – Applications of Forces (Part A Equilibrium and Statics of Particle) By the end of the sub-unit, students should: • understand that a body is in equilibrium under a set of concurrent (acting through the same point) forces is if their resultant is zero; • know that vectors representing forces in equilibrium form a closed polygon; • understand how to solve problems involving equilibrium of a particle under coplanar forces, including particles on inclined planes and 2D vectors; • be able to solve statics problems for a system of forces which are not concurrent (e.g. ladder problems), thus applying the principle of moments for forces at any angle.	This topic is a natural extension of AS Mathematics - Mechanics content (see SoW Unit 8a), which considers statics for systems whose forces are perpendicular (and do not need resolving at any angle) and $\mathbf{i}$,$\mathbf{j}$ vector examples. Recall the previous definition of equilibrium: the vector sum of the forces is zero, so the sum of their resolved parts in any direction is zero. The book on an inclined plane provides the most common example of a weight on a slope. Stress the importance of key phrases like 'rough plane', which will introduce a frictional force. Also highlight the part of the sentence that says 'the book is *on the point* of moving *down* the plane' and emphasise that this indicates that the frictional force is in the up direction and is at its limiting value. Cover examples • Where the angle of incline is given in arctan or arcsin form, so students have to construct and read off sin and cos of the angle. • Where weights are held in equilibrium by two strings at any angle (this is the same as a weight being tied onto a particular point of a single string - the knot makes it effectively two pieces of string with two different tensions).You could show an alternative graphical solution. For example, combining. the three forces to form a closed triangle (equilibrium means no resultant). Applying the sine rule to this triangle gives a useful result called Lami's theorem, but it can only be used for three forces in equilibrium. • Where a ring is free to slide on a string (hence one tension). • Where the forces are given in terms of $\mathbf{i}$ and $\mathbf{j}$. Finally, move on to ladder-type problems which will revise moments and then extend to any angle, as the forces will not be concurrent. Extend the moments formula to '*perpendicular force* × *distance*' and resolve the force to find its component at right angles to the full distance from the moments point. Show students how to use the alternative formula '*force* × *perpendicular distance*', by measuring the perpendicular distance from the moments point to the line of action of the force. Also make sure that students are clear about the directions of the frictional force (for examples involving rough surfaces) and the reactions at the wall and ground being labelled differently.
Chapter 7 – Applications of Forces (Part B Dynamics of a Particle) By the end of the sub-unit, students should: • know and understand the meaning of Newton's second law; • be able to formulate the equation of motion for a particle in 1-dimensional motion where the resultant force is mass × acceleration; • be able to formulate the equation of motion for a particle in 2-dimensional motion where the resultant force is mass × acceleration; • be able to formulate and solve separate equations of motion for connected particles, where one of the particles could be on an inclined and/or rough plane.	This topic is a natural extension of AS Mathematics - Mechanics content (see SoW Unit 8a), which considers dynamics for systems whose forces are perpendicular (and do not need resolving at any angle) and $\mathbf{i}$, $\mathbf{j}$ vector examples. Recall the previous definition of dynamics: the *vector sum of the forces* = *mass* × *acceleration*, so the sum of their resolved parts in any direction can now be represented as a *single* force. This force is called the resultant and is equal to *mass* × *acceleration* (Newton's second law). We can use the equations of motion for constant acceleration to describe the motion in more detail e.g. time taken to come to rest etc. The basic mathematical modelling is identical to that of setting up a statics problem, except when you resolve in the direction of motion; there will be a 'winning' resultant force. For inclined plane problems stress, that it is often easier, to resolve along and perpendicular to the plane. Some students find it hard to understand that even though the particle is moving up/down, the forces are 'balanced' if we resolve perpendicular to the plane. Make sure you cover examples in which a force 'pushing' up the plane is removed at a certain point. This means the frictional force and component of weight now influence the subsequent motion and act as 'braking forces' causing a retardation, bringing the particle to instantaneous rest (and then the friction changes direction, as the particle wants to slide back down the plane). Provide some examples where the forces are given in terms of $\mathbf{i}$ and $\mathbf{j}$. These are solved by applying Newton's Second Law in vector form, hence $\boldsymbol{F} = m\boldsymbol{a}$. Connected particle problems (previously covered in AS Mathematics - Mechanics content, see SoW Unit 8b) can now be extended so at least one of the particles is placed on a rough or smooth inclined plane and/or a rough horizontal plane. This introduces the resolving and frictional concepts from the previous unit. For 'car and caravan' type questions, the tow rope or tow-bar can now be modelled at an angle rather than horizontally.

Chapter/Objectives	Teaching Points

FURTHER KINEMATICS

Chapter 8 – Further Kinematics (Part A Constant Acceleration in 2D) By the end of the sub-unit, students should: • be able to recognise when the use of constant acceleration formulae is appropriate; • be able to write positions, velocities and accelerations in vector form; • understand the language of kinematics appropriate to motion in 2 dimensions • be able to find the magnitude and direction of vectors; • be able to extend techniques for motion in 1 dimension to 2 dimensions by using vectors; • know how to use velocity triangles to solve simple problems; • understand and use *suvat* formulae for constant acceleration in 2D; • know how to apply the equations of motion to **i**, **j** vector problems; • be able to use $\boldsymbol{v} = \boldsymbol{u} + \boldsymbol{a}t$, $\boldsymbol{r} = \boldsymbol{u}t + \frac{1}{2}\boldsymbol{a}t^2$ etc. with vectors given in **i** , **j** or column vector form.	This topic enables us to use the familiar *suvat* formulae for constant acceleration for more complex motions in two dimensions. It is important to stress that the acceleration may have different values for the **i** and **j** components, but is fixed in value for that direction and is therefore constant. Illustrate this by reviewing projectile motion (covered in Unit 6), which showed that the acceleration was zero in the horizontal direction and ±9.8 m s^{-2} in the vertical direction, hence for a full trajectory $\boldsymbol{a} = (0\mathbf{i} - 9.8\mathbf{j})$ m s^{-2}. This gives a curved (parabolic) path even though the accelerations are constant. Cover examples which ask for the speed, distance and direction of motion. Make sure that students can pick out the keywords, and realise when the answer can be left in **i**,**j** form and when to form a triangle and use Pythagoras and tan to calculate the magnitude and direction (e.g. when asked for the speed and direction of motion of a particle). Also stress that the *angle* of the velocity vector gives the true direction of motion and that the acceleration's magnitude does not have a special keyword, but will just be asked for as magnitude of the acceleration.

Chapter/Objectives	Teaching Points
Chapter 8 – Further Kinematics (Part B Variable Acceleration using Calculus) By the end of the sub-unit, students should: • be able to extend techniques for motion in 1 dimension to 2 dimensions by using calculus and vector versions of equations for variable force/acceleration problems; • understand the language and notation of kinematics appropriate to variable motion in 2 dimensions, i.e. knowing the notation $\dot{r}$ and $\ddot{r}$ for variable acceleration in terms of time.	This topic links directly to, and is an extension of AS Mathematics - Mechanics content (see SoW Unit 9), which used: $v = \frac{\mathrm{d}s}{\mathrm{d}t}$, $a = \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}^2 s}{\mathrm{d}t^2}$ and $s = \int v\mathrm{d}t$, $v = \int a\mathrm{d}t$ to model the rates of change for motion of a particle subject to a variable force. Motions can now be more complicated as the forces in the **i** and **j** directions can differ and be variable (i.e. ***F*** = *m**a***). Also the notation for 2D motion replaces the displacement, s, with position vector, ***r***. Velocity, ***v***, can be defined as $\dot{r}$ and the acceleration vector can be called $\ddot{r}$ (rather than ***a***). Introduce this notation to students, explaining how the dot above the ***r*** denotes how many times the ***r*** has been differentiated with respect to time. Hence $\ddot{r}$ (representing the acceleration) effectively means ***r*** differentiated *twice* with respect to time or $\frac{\mathrm{d}^2 r}{\mathrm{d}t^2}$. The other vital point to stress is when we integrate $\dot{r}$ (or ***v***) to obtain the displacement ***r***, we have to introduce a vector constant of integration in the form c**i** + k**j** (rather than just + c). Any conditions provided in the question (e.g. the particle is initially at the point with position vector (3**i** + 2**j)** m) allow us to substitute into the expression for ***r*** and calculate the constants. Ask questions along the lines of: Consider an aeroplane taking off. Its position is given by ***r*** = (80t**i** + 0.5t^3 **j**)m. What is its velocity and acceleration at time t? Now criticise the model. (Hint: consider motion in the x-direction) Reverse the process: a particle has acceleration ***a*** = (4*t***i** + 2**j**) m s^{-2} and is initially at the origin moving with velocity 2**i** m s^{-1}. Find $\dot{r}$ and **r** using integration. (Be careful with the constants of integration!) Just as in the 1-dimensional case, we do not need to use calculus every time; if the acceleration vector is constant, we can use vector forms of the *suvat* formulae as in Unit 8a. Questions on this topic often ask about the direction of motion: stress that this is given by the direction of the velocity vector. To find when an object is moving due North, the East component of the velocity vector is zero and the North component positive. Finally, a question may ask for the force acting on the particle of mass *m* kg. In this situation students will need to find the acceleration ($\ddot{r}$) at time *t* and then state the force ***F*** as ***F*** = $m\ddot{r}$ or ***F*** = *m**a*** (in terms of **i** and **j**).