Year 12 Induction Week: July 2020

Further Mathematics Induction Work

Complex Numbers

Complex numbers arise when you attempt to square root a negative number. The letter i is used to represent the square root of negative 1. Using your knowledge of surds, it is possible to write the square root of any negative number in terms of i.

Example 1: Write the square root of -50 in terms of i.
$\sqrt{-50} = \sqrt{-1}\ \sqrt{50} = i\ \sqrt{25}\ \sqrt{2} = (5\sqrt{2})i$

You can also add and subtract complex numbers by considering the real and imaginary parts. When a complex number is in the form a + bi where a and b are real numbers, the first half of this expression "a" is known as the real part of the complex number, the second half of this expression "bi" is known as the imaginary part of the complex number

Example 2: Simplify (2 + 3i) + (5 – i)
Simply collect the like terms for the real and imaginary parts to give 2 + 5 + 3i – i = 7 + 2i

Similarly, using your knowledge of surds, consider $i^2 = \sqrt{-1} \times \sqrt{-1} = -1$

Example 3: Simplify (2 + 3i) (5 – i)
Here you need to multiply the terms out as you are used to seeing for quadratic equations
2x5 + 2x-i + 3i x 5 + 3i x –i
$10 - 2i + 15i - 3i^2$
Using the result above for i^2 gives
10 -2i + 15i +3
13 + 13i

The next page has a number of questions to work through, refer to the examples above

Exercise 1

Simplify, giving your answer in the form $a + b\mathrm{i}$, where $a \in \mathbb{R}$ and $b \in \mathbb{R}$.

1 $(5 + 2\mathrm{i}) + (8 + 9\mathrm{i})$

2 $(4 + 10\mathrm{i}) + (1 - 8\mathrm{i})$

3 $(7 + 6\mathrm{i}) + (-3 - 5\mathrm{i})$

4 $(2 - \mathrm{i}) + (11 + 2\mathrm{i})$

5 $(3 - 7\mathrm{i}) + (-6 + 7\mathrm{i})$

6 $(20 + 12\mathrm{i}) - (11 + 3\mathrm{i})$

7 $(9 + 6\mathrm{i}) - (8 + 10\mathrm{i})$

8 $(2 - \mathrm{i}) - (-5 + 3\mathrm{i})$

9 $(-4 - 6\mathrm{i}) - (-8 - 8\mathrm{i})$

10 $(-1 + 5\mathrm{i}) - (-1 + \mathrm{i})$

11 $(3 + 4\mathrm{i}) + (4 + 5\mathrm{i}) + (5 + 6\mathrm{i})$

12 $(-2 - 7\mathrm{i}) + (1 + 3\mathrm{i}) - (-12 + \mathrm{i})$

13 $(18 + 5\mathrm{i}) - (15 - 2\mathrm{i}) - (3 + 7\mathrm{i})$

14 $2(7 + 2\mathrm{i})$

15 $3(8 - 4\mathrm{i})$

16 $7(1 - 3\mathrm{i})$

17 $2(3 + \mathrm{i}) + 3(2 + \mathrm{i})$

18 $5(4 + 3\mathrm{i}) - 4(-1 + 2\mathrm{i})$

19 $\left(\frac{1}{2} + \frac{1}{3}\mathrm{i}\right) + \left(\frac{5}{2} + \frac{5}{3}\mathrm{i}\right)$

20 $(3\sqrt{2} + \mathrm{i}) - (\sqrt{2} - \mathrm{i})$

Write in the form $b\mathrm{i}$, where $b \in \mathbb{R}$.

21 $\sqrt{(-9)}$

22 $\sqrt{(-49)}$

23 $\sqrt{(-121)}$

24 $\sqrt{(-10\,000)}$

25 $\sqrt{(-225)}$

26 $\sqrt{(-5)}$

27 $\sqrt{(-12)}$

28 $\sqrt{(-45)}$

29 $\sqrt{(-200)}$

30 $\sqrt{(-147)}$

Solve these equations.

31 $x^2 + 2x + 5 = 0$

32 $x^2 - 2x + 10 = 0$

33 $x^2 + 4x + 29 = 0$

34 $x^2 + 10x + 26 = 0$

35 $x^2 - 6x + 18 = 0$

36 $x^2 + 4x + 7 = 0$

37 $x^2 - 6x + 11 = 0$

38 $x^2 - 2x + 25 = 0$

39 $x^2 + 5x + 25 = 0$

40 $x^2 + 3x + 5 = 0$

Exercise 2

Simplify these, giving your answer in the form $a + bi$.

1 $(5 + i)(3 + 4i)$

2 $(6 + 3i)(7 + 2i)$

3 $(5 - 2i)(1 + 5i)$

4 $(13 - 3i)(2 - 8i)$

5 $(-3 - i)(4 + 7i)$

6 $(8 + 5i)^2$

7 $(2 - 9i)^2$

8 $(1 + i)(2 + i)(3 + i)$

9 $(3 - 2i)(5 + i)(4 - 2i)$

10 $(2 + 3i)^3$

Simplify.

11 i^6

12 $(3i)^4$

13 $i^5 + i$

14 $(4i)^3 - 4i^3$

15 $(1 + i)^8$

Exercise 3

1 $a + 2b + 2ai = 4 + 6i$, where a and b are real.
Find the value of a and the value of b.

2 $(a - b) + (a + b)i = 9 + 5i$, where a and b are real.
Find the value of a and the value of b.

3 $(a + b)(2 + i) = b + 1 + (10 + 2a)i$, where a and b are real.
Find the value of a and the value of b.

4 $(a + i)^3 = 18 + 26i$, where a is real.
Find the value of a.

5 $abi = 3a - b + 12i$, where a and b are real.
Find the value of a and the value of b.

6 Find the real numbers x and y, given that

$$\frac{1}{x + iy} = 3 - 2i$$

7 Find the real numbers x and y, given that

$$(x + iy)(1 + i) = 2 + i$$

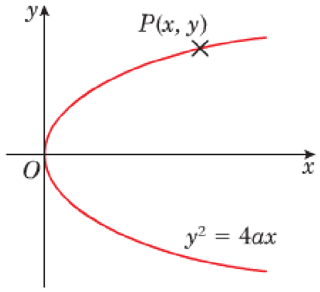
Coordinate Systems

The Parabola

- The curve opposite is an example of a **parabola** which has parametric equations:

  $x = at^2, y = 2at, t \in \mathbb{R}$,

  where a is a positive constant.



- The Cartesian equation of this curve is $y^2 = 4ax$ where a is a positive constant.
- This curve is symmetrical about the x-axis.
- A general point P on this curve has coordinates $P(x, y)$ or $P(at^2, 2at)$.

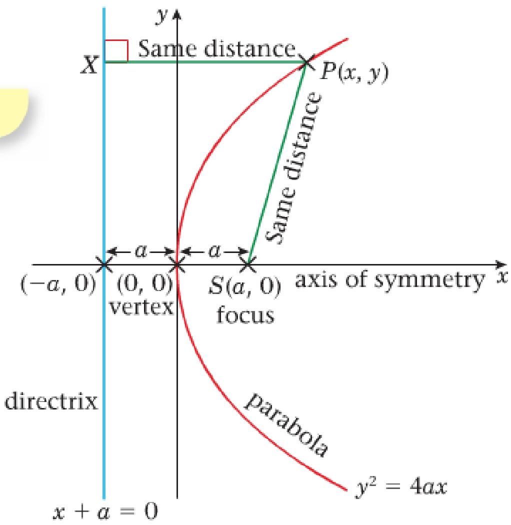
A *locus of points* is a set of points which obey a certain rule.

- A parabola is the ***locus of points*** where every point $P(x, y)$ on the parabola is the same distance from a fixed point S, called the focus, and a fixed straight line called the directrix.
- The parabola is the set of points where $SP = PX$.

  The **focus**, S, has coordinates $(a, 0)$

  The **directrix** has equation $x + a = 0$.

  The **vertex** is at the point $(0, 0)$.



Exercise 1

1 Find an equation of the parabola with

a focus (5, 0) and directrix $x + 5 = 0$,

b focus (8, 0) and directrix $x + 8 = 0$,

c focus (1, 0) and directrix $x = -1$,

d focus $\left(\frac{3}{2}, 0\right)$ and directrix $x = -\frac{3}{2}$,

e focus $\left(\frac{\sqrt{3}}{2}, 0\right)$ and directrix $x + \frac{\sqrt{3}}{2} = 0$.

2 Find the coordinates of the focus, and an equation for the directrix of a parabola with these equations.

a $y^2 = 12x$

b $y^2 = 20x$

c $y^2 = 10x$

d $y^2 = 4\sqrt{3}x$

e $y^2 = \sqrt{2}x$

f $y^2 = 5\sqrt{2}x$

3 A point $P(x, y)$ obeys a rule such that the distance of P to the point (3, 0) is the same as the distance of P to the straight line $x + 3 = 0$. Prove that the locus of P has an equation of the form $y^2 = 4ax$, stating the value of the constant a.

4 A point $P(x, y)$ obeys a rule such that the distance of P to the point $(2\sqrt{5}, 0)$ is the same as the distance of P to the straight line $x = -2\sqrt{5}$. Prove that the locus of P has an equation of the form $y^2 = 4ax$, stating the value of the constant a.

Exercise 2

1. A point P(x, y) obeys a rule such that the distance of P to the point (5, 0) is the same as the distance of P to the straight line $x = -5$. Find an equation of the locus of P.

2. A point P(x, y) obeys a rule such that the distance of P to the point (a, 0) is the same as the distance of P to the straight line $x + a = 0$ Find an equation of the locus of P

3. A point P(x, y) obeys a rule such that the distance of P to the point (2, 1) is the same as the distance of P to the straight line $x + 4 = 0$. Find an equation of the locus of P.

4. The point P (8, –8) lies on the parabola C with equation $y^2 = 8x$. The point S is the focus of the parabola. The line l passes through S and P.
   (a) Find the coordinates of S
   (b) Find an equation for l, giving your answer in the form $ax + by + c = 0$, where a, b and c are integers.
   The line l meets the parabola C at the point Q. The point M is the mid-point of PQ.
   (c) Find the coordinates of Q.
   (d) Find the coordinates of M.
   (e) Draw a sketch showing parabola C, the line l and the points P, Q, S and M

Matrices

When you add and subtract matrices, you can simply apply the rules to each individual cell. So for example the top left plus the top left. When you multiply by a scalar, apply this to every term in the matrix

Example 1: 2 (3 1) + (4 5) = (6 2) + (4 5) = (10 7)

Exercise 1

1 For the matrices $\mathbf{A} = \begin{pmatrix} 2 & 0 \\ 4 & -6 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, find

a $3\mathbf{A}$ **b** $\frac{1}{2}\mathbf{A}$ **c** $2\mathbf{B}$.

2 Find the value of k and the value of x so that $\begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix} + k\begin{pmatrix} 0 & 2 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 7 \\ x & 0 \end{pmatrix}$.

3 Find the values of a, b, c and d so that $2\begin{pmatrix} a & 0 \\ 1 & b \end{pmatrix} - 3\begin{pmatrix} 1 & c \\ d & -1 \end{pmatrix} = \begin{pmatrix} 3 & 3 \\ -4 & -4 \end{pmatrix}$.

4 Find the values of a, b, c and d so that $\begin{pmatrix} 5 & a \\ b & 0 \end{pmatrix} - 2\begin{pmatrix} c & 2 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 9 & 1 \\ 3 & d \end{pmatrix}$.

5 Find the value of k so that $\begin{pmatrix} -3 \\ k \end{pmatrix} + k\begin{pmatrix} 2k \\ 2k \end{pmatrix} = \begin{pmatrix} k \\ 6 \end{pmatrix}$.

- **Whilst the rule for adding and subtracting matrices appears to be quite natural, it is the rule for multiplying matrices that gives them their useful properties.**
- **The basic operation consists of multiplying each element in the row of the left hand matrix by each corresponding element in the column of the right hand matrix and adding the results together.**
- **The number of columns in the left hand matrix must equal the number of rows in the right hand matrix.**
- **The product will then have the same number of rows as the left hand matrix and the same number of columns as the right hand matrix.**

So if

$$\mathbf{A} \times \mathbf{B} = \mathbf{C}$$

Dimensions: $(n \times m) \times (m \times k)$ $(n \times k)$

n is from the number of rows in **A**.
k is from the number of columns in **B**.

These numbers must be the same.

Exercise 2

1 Given the dimensions of the following matrices:

Matrix	**A**	**B**	**C**	**D**	**E**
Dimension	2×2	1×2	1×3	3×2	2×3

Give the dimensions of these matrix products.

a **BA** **b** **DE** **c** **CD**

d **ED** **e** **AE** **f** **DA**

2 Find these products.

a $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} -1 \\ 2 \end{pmatrix}$ **b** $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 0 & 5 \\ -1 & -2 \end{pmatrix}$

3 The matrix $\mathbf{A} = \begin{pmatrix} -1 & -2 \\ 0 & 3 \end{pmatrix}$ and the matrix $\mathbf{B} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$.

Find

a **AB** **b** $\mathbf{A}^2$

$\mathbf{A}^2$ means $\mathbf{A} \times \mathbf{A}$

4 The matrices **A**, **B** and **C** are given by

$\mathbf{A} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 3 & 1 \\ -1 & 2 \end{pmatrix}$, $\mathbf{C} = \begin{pmatrix} -3 & -2 \end{pmatrix}$.

Determine whether or not the following products are possible and find the products of those that are.

a **AB** **b** **AC** **c** **BC**

d **BA** **e** **CA** **f** **CB**

5 Find in terms of a $\begin{pmatrix} 2 & a \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 3 & 0 \\ 0 & -1 & 2 \end{pmatrix}$.

6 Find in terms of x $\begin{pmatrix} 3 & 2 \\ -1 & x \end{pmatrix}\begin{pmatrix} x & -2 \\ 1 & 3 \end{pmatrix}$.

7 The matrix $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$.

Find

a $\mathbf{A}^2$

b $\mathbf{A}^3$

c Suggest a form for $\mathbf{A}^k$.

You might be asked to prove this formula for $\mathbf{A}^k$ in FP1 using induction from Chapter 6.

8 The matrix $\mathbf{A} = \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix}$.

a Find, in terms of a and b, the matrix $\mathbf{A}^2$.

Given that $\mathbf{A}^2 = 3\mathbf{A}$

b find the value of a.

- If $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then $\mathbf{A}^{-1} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

  and then $\mathbf{AA}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{I}$

The value of $ad - bc$ is called the **determinant** of **A** and written det(**A**).

- $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $\det(\mathbf{A}) = ad - bc$ so $\mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

Notice that if $\det(\mathbf{A}) = 0$ you will not be able to find $\mathbf{A}^{-1}$ because $\frac{1}{\det(\mathbf{A})}$ is not defined, in such cases we say **A** is **singular**.

- If det(**A**) = 0, then **A** is a **singular matrix** and $\mathbf{A}^{-1}$ cannot be found.
  If det(**A**) $\neq$ 0, then **A** is a **non-singular matrix** and $\mathbf{A}^{-1}$ exists.

Exercise 3

1 Use inverse matrices to solve the following simultaneous equations

a $7x + 3y = 6$
$-5x - 2y = -5$

b $4x - y = -1$
$-2x + 3y = 8$

2 Use inverse matrices to solve the following simultaneous equations

a $4x - y = 11$
$3x + 2y = 0$

b $5x + 2y = 3$
$3x + 4y = 13$

Series

- The formula for the sum of an arithmetic series is

$$S_n = \frac{n}{2}[2a + (n-1)d]$$

or $$S_n = \frac{n}{2}(a + L)$$

where a is the first term, d is the common difference, n is the number of terms and L is the last term in the series.

You could be asked to prove these formulae.

- The general rule for the sum of a geometric series is $S_n = \dfrac{a(r^n - 1)}{r - 1}$ or $\dfrac{a(1 - r^n)}{1 - r}$

For example:

$\sum_{n=1}^{10} 2n$ means sum of $2n$ from $n = 1$ to $n = 10$

$$= 2 + 4 + 6 + 8 + 10 + 12 + 14 + 16 + 18 + 20$$

$$\sum_{n=1}^{10} U_n = U_1 + U_2 + U_3 + \ldots + U_{10}$$

$\sum_{r=0}^{10} (2 + 3r)$ means the sum of $2 + 3r$ from $r = 0$ to $r = 10$

$$= 2 + 5 + 8 + \ldots + 32$$

$\sum_{r=5}^{r=15} (10 - 2r)$ means the sum of $(10 - 2r)$ from $r = 5$ to $r = 15$

$$= 0 + -2 + -4 + \ldots + -20$$

Exercise 1

1 Rewrite the following sums using Σ notation:

a $4 + 7 + 10 + \ldots + 31$

b $2 + 5 + 8 + 11 + \ldots + 89$

c $40 + 36 + 32 + \ldots + 0$

d The multiples of 6 less than 100

2 Calculate the following:

a $\sum_{r=1}^{5} 3r$

b $\sum_{r=1}^{10} (4r - 1)$

c $\sum_{r=1}^{20} (5r - 2)$

d $\sum_{r=0}^{5} r(r + 1)$

3 For what value of n does $\sum_{r=1}^{n} (5r + 3)$ first exceed 1000?

4 For what value of n would $\sum_{r=1}^{n} (100 - 4r) = 0$?

Exercise 2

1 Write out each of the following as a sum of terms, and hence calculate the sum of the series.

a $\sum_{r=1}^{10} r$ **b** $\sum_{p=3}^{8} p^2$ **c** $\sum_{r=1}^{10} r^3$

d $\sum_{p=1}^{10} (2p^2 + 3)$ **e** $\sum_{r=0}^{5} (7r + 1)^2$ **f** $\sum_{i=1}^{4} 2i(3 - 4i^2)$

2 Write each of the following as a sum of terms, showing the first three terms and the last term.

a $\sum_{r=1}^{n} (7r - 1)$ **b** $\sum_{r=1}^{n} (2r^3 + 1)$

c $\sum^{n} (j - 4)(j + 4)$ **d** $\sum^{k} p(p + 3)$

3 In each part of this question write out, as a sum of terms, the two series defined by $\sum f(r)$; for example, in part **c**, write out the series $\sum_{r=1}^{10} r^2$ and $\sum_{r=1}^{10} r$. Hence, state whether the given statements relating their sums are true or not.

a $\sum_{r=1}^{n} (3r + 1) = \sum_{r=2}^{n+1} (3r - 2)$ **b** $\sum_{r=1}^{n} 2r = \sum_{r=0}^{n} 2r$ **c** $\sum_{r=1}^{10} r^2 = \left(\sum_{1}^{10} r\right)^2$

d $\sum_{r=1}^{4} r^3 = \left(\sum_{r=1}^{4} r\right)^2$ **e** $\sum_{r=1}^{n} (3r^2 + 4) = 3\sum_{r=1}^{n} r^2 + 4$

4 Express these series using $\sum$ notation.

a $3 + 4 + 5 + 6 + 7 + 8 + 9 + 10$

b $1 + 8 + 27 + 64 + 125 + 216 + 343 + 512$

c $11 + 21 + 35 + ... + (2n^2 + 3)$

d $11 + 21 + 35 + ... + (2n^2 - 4n + 5)$

e $3 \times 5 + 5 \times 7 + 7 \times 9 + ... + (2r - 1)(2r + 1) + ...$ to k terms.

The sums of powers, k, of the first n natural numbers, for the cases $k = 0, 1, 2$ and 3 are

- $\sum_{r=1}^{n} r^0 = \sum_{r=1}^{n} 1 = 1 + 1 + 1 ... + 1 = n$
- $\sum_{r=1}^{n} r = 1 + 2 + 3 + ... + n = \frac{n}{2}(n + 1)$
- $\sum_{r=1}^{n} r^2 = 1^2 + 2^2 + 3^2 + ... + n^2 = \frac{n}{6}(n + 1)(2n + 1)$
- $\sum_{r=1}^{n} r^3 = 1^3 + 2^3 + 3^3 + ... + n^3 = \frac{n^2}{4}(n + 1)^2$

These should be learnt.

Note:
$\sum_{r=1}^{n} r^3 = \left(\sum_{r=1}^{n} r\right)^2$

Exercise 3

1 Verify that $\sum_{r=1}^{n} r^2 = \frac{n}{6}(n+1)(2n+1)$ is true for $n = 1$, 2 and 3.

2 **a** By writing out each series, evaluate $\sum_{r=1}^{n} r$ for $n = 1, 2, 3$ and 4.

b By writing out each series, evaluate $\sum_{r=1}^{n} r^3$ for $n = 1, 2, 3$ and 4.

c What do you notice about the corresponding results for each value of n ?

3 Using the appropriate formula, evaluate **a** $\sum_{r=1}^{100} r^2$ **b** $\sum_{r=20}^{40} r^2$ **c** $\sum_{r=1}^{30} r^3$ **d** $\sum_{r=25}^{45} r^3$

4 Use the formula for $\sum_{r=1}^{n} r^2$ or $\sum_{r=1}^{n} r^3$ to find the sum of

a $1^2 + 2^2 + 3^2 + 4^2 + \ldots + 52^2$

b $2^3 + 3^3 + 4^3 + \ldots + 40^3$

c $26^2 + 27^2 + 28^2 + 29^2 + \ldots + 100^2$

d $1^2 + 2^2 + 3^2 + \ldots + (k+1)^2$

e $1^3 + 2^3 + 3^3 + \ldots + (2n-1)^3$