Boolean Algebra Simplifications

The concept of Boolean Algebra demonstrations is to, in essence, reduce the number of inputs required to obtain a working solution. The advantages are many but if you think of how a processor works by simplifying the number of instructions you reduce the number of RAM slots being used therefore making both the program you are using and the processor run more efficiently.

The “art” behind the simplification of Boolean Algebra expressions is based on the use of the Formulas that you can find on the website. I will need your patience and understanding on this as they are not the easiest thing to follow. Bear in mind that this is usually a year 1 Logic subject in a university course. I will always prepare you for a more difficult and challenging Boolean Algebra simplification than you will find in an exam condition!

Some of the expressions on the formula sheet require no explanation or indeed demonstration. I have attached the reasoning to them here:

$$0 + 0 = 0 \qquad 0 . 0 = 0$$
$$0 + 1 = 1 \qquad 0 . 1 = 0$$
$$1 + 0 = 1 \qquad 1 . 0 = 0$$
$$1 + 1 = 1 \qquad 1 . 1 = 1$$

None of the above require any type of explanation; these are straightforward and very obvious!

Because of the reasoning above the first 8 rules are derived without demonstration, remember that A being an atomic sentence can only be Zero (0) / False or One (1) / True

Rule 9 is the double negation or known as double complement

$$\overline{\overline{X}} = X$$

This is a simple concept and comes from negating a statement twice meaning the negation of a statement happening twice means the statement stands! Bit of a tongue twister!

Rules 10 to 14 are all from basic Algebra, so they should not come as a surprise! Commutative, associative and distribution are all basic mathematical concepts. You learned these back in primary school but chances are someone never told you the right names for it!

Now we come to some rules that do need demonstration!

The way we are going to demonstrate the laws is a version of algebraic demonstration and it is the one I would like you to get used to. Technically you pick either the right hand side or the left hand side and arrive to the opposite side!

Rule 15

$X + Y.Z = (X + Y).(X + Z)$ Dual of Distributive Law

$$
\begin{aligned}
(X + Y)(X + Z) &= XX + XZ + YX + YZ \\
&= X + XZ + YX + YZ, && X.X = X \\
&= X(1 + Z) + YX + YZ \\
&= X + YX + YZ, && 1 + Z = 1 \\
&= X(1 + Y) + YZ \\
&= X + YZ && 1 + Y = 1
\end{aligned}
$$

Justifications are welcomed but not necessary in exam conditions!

Rule 16

$X + XZ = X$

$$
\begin{aligned}
&X + XZ = X \\
&\text{L.H.S.} = X + XZ = X(1 + Z) = X.1 = X, \qquad I + Z = 1 \\
&\qquad = \text{R.H.S.}
\end{aligned}
$$

Rule 17

$X(X + Z) = X$

$$
\begin{aligned}
&X(X + Z) = X \\
&\text{L.H.S.} = X(X + Z) \\
&\qquad = X X + XZ && \text{By distributive law} \\
&\qquad = X + XZ, && \text{as } X.X = X \\
&\qquad = X(1 + Z), && \text{As } 1 + Z = 1 \\
&\qquad = X.1 \\
&\qquad = X \\
&\text{L.H.S.} = \text{R.H.S.}
\end{aligned}
$$

Rule 18

$X + \overline{X}Y = X + Y$

L.H.S. $= X + \overline{X}\ Y = (X + \overline{X}) . (X + Y)$ By rule 15 dual Of distributive law.

$= 1 . (X + Y)$ as $X + \overline{X} = 1$

$= X + Y$

L.H.S. = R.H.S.

Rule 19

$X(\overline{X} + Y) = X.Y$

$X.(\overline{X} + Y) = X . Y$

L.H.S. $= X(\overline{X} + Y) = X\overline{X} + XY$ By distributive law

$= 0 + XY$ as $X . \overline{X} = 0$

$= XY$

L.H.S. = R.H.S.

For the De Morgan's laws demonstration this requires a full algebraic demonstration for both the ***absorbing element in Logic (0)*** and the ***neutral element in Logic (1)***. Usually this requires you to have full knowledge of advanced demonstration skills which are beyond even what A-Level maths will do and we in Computing end up doing more advanced concepts than they do. For that reason alone I will provide the De Morgan's laws proof done by induction, in this case, induction means the use of a truth table!

Rule 20

$\overline{X+Y} = \overline{X} . \overline{Y}$

X	Y	$\overline{X}$	$\overline{Y}$	X + Y	$\overline{X+Y}$	$\overline{X} . \overline{Y}$
0	0	1	1	0	1	1
0	1	1	0	1	0	0
1	0	0	1	1	0	0
1	1	0	0	1	0	0

L.H.S. =R.H.S.

Rule 21

$\overline{X.Y} = \overline{X} + \overline{Y}$

X	Y	$\overline{X}$	$\overline{Y}$	X .Y	$\overline{X . Y}$	$\overline{X} + \overline{Y}$
0	0	1	1	0	1	1
0	1	1	0	0	1	1
1	0	0	1	0	1	1
1	1	0	0	1	0	0

L.H.S. = R.H.S.

Are there more rules? Yes there are a lot more but the board is only concerned with the easy ones which are all of the above! Mind you De Morgan's laws always appear!

Examples:

1.

Express the Boolean function

$XY + YZ + \overline{Y}Z = XY + Z$

$$
\begin{aligned}
\text{L.H.S.} &= XY + YZ + \overline{Y}Z \\
&= XY + Z(Y + \overline{Y}) \\
&= XY + Z.1 \\
&= XY + Z \\
\text{L.H.S} &= \text{R.H.S.}
\end{aligned}
$$

2.

Simplify the Boolean expressions:

(i) $(X + Y)(X + \overline{Y})(\overline{X} + Z)$

(ii) $XYZ + X\overline{Y}Z + XY\overline{Z}$

(i) First simplify $(X + Y)(X + Y)$

$(X + Y)(X + \overline{Y}) = XX + X\overline{Y} + YX + Y\overline{Y}$

$= X + X\overline{Y} + YX + O,$ as $XX = X$

as $Y\overline{Y} = 0$

$= X + X(\overline{Y} + Y),$ as $\overline{Y} + Y = 1$

$= X + X \cdot 1,$ as $X \cdot 1 = X$

$= X + X$

$= X$

Now $(X + Y)(X + \overline{Y})(\overline{X} + Z)$

$= X(\overline{X} + Z)$

$= X\overline{X} + XZ,$ by distributive law

$= 0 + XZ$

$= XZ$

(ii) $XYZ + X\overline{Y}Z + XY\overline{Z}$

$= XZ(Y + \overline{Y}) + XY\overline{Z}$

$= XZ + XY\overline{Z},$ as $Y + \overline{Y} = 1$

$= X(Z + Y\overline{Z})$

$= X[(Z + Y).(Z + \overline{Z})]$, (By Rule 15 dual of distributive

$= X[(Z + Y). 1] = X(Z + Y)$

$= X(Y + Z),$ by commutative law.